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NEW SERIES

OF THE

MATHEMATICAL

REPOSITORY.

BY THOMAS LEYBOURN,
Of the Royal Military College.

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II. The first and second parts of the Vol. must follow in their order. The two sets of questions of Vol. 5 (viz. the signatures A and B) which are stitched up with the numbers forming the fourth Vol. must be preserved apart till the fifth Vol. is completed.

III. Cambridge Problems to be placed after the second part.

THE

MATHEMATICAL REPOSITORY,

MATHEMATICAL QUESTIONS.

To be answered in Number XVI.

I. QUESTION 371, *by the Rev Mr.* W. Wood.

Given to find x and y the two equations, $3x^2 - y^2 = a$, and $x^3 - y^3 - 2xy^2 = b$.

II. QUESTION 372, *by* B. A.

Let C be the circumference of a circle, D the diameter, c the chord of any arch a;

$$\text{then } \frac{4a\text{D}\,(\text{C} - a)}{\frac{3}{4}\text{C}^2 - (\text{C} - a)\,a} = c, \text{ nearly.}$$

Required the investigation?

III. QUESTION 373, *by* Z.

Find what conditions must have place among the co-efficients of $x^3 - px^2 + qx - r = 0$ that the roots may be in harmonical progression, and find those roots.

IV. QUESTION 374, *by* Julius.

Exponential equations of the form $x^x = a$, may be divided into three classes, viz. those having only one real root, those having two real roots, and those which have no real root. It is required to point out the limits, and in the case of two real roots, to shew what functions they are of each other?

V. QUESTION 375, *by Mr.* Cunliffe, *Royal Military College.*

It is required to find two such rational fractions, that the cube of either being added to the square of the other, shall make the same sum; and furthermore, that their sum and sum of their squares may both be square numbers.

VI. QUESTION 376, *by Mr.* Cunliffe.

Find the equation of the curve which is the locus of the intersection of the diagonals of a trapezium whose sides are given in length, and one of the sides given by position.

VII. QUESTION 377, *by Mr.* Cunliffe.

What is the relation of the diameters of the 3 circles, passing through the extremities of the sides, and point of intersection of the perpendiculars from the angles upon the sides of a plane triangle. — Equal — T. F

VIII. QUESTION 378, *by Mr.* Cunliffe.

Given the base, the line from the vertical angle to the middle of the base, and the line bisecting the vertical angle and terminating in the base; to construct the plane triangle.

IX. QUESTION 379, *by* Palaba.

Given that the distance of the centre of gravity of an area from its vertex is an nth part of the abscissa; to find the distance of the centre of gravity of the solid generated by the same area revolving round its axis.

X. QUESTION 380, *by* Palaba.

Find the distance from the vertex of the centre of gravity of the area of the catenaria without the aid of logarithms.

XI. QUESTION 381, *by* PALABA.

The equation to the lemniscata being $(x^2+y^2)^2 = x^2 - y^2$; find its area contained between the values of x, 0 and 1.

XII. QUESTION 382, *by* PALABA.

Determine that point in a curve whose equation is $a^{n-1}x = y^n$ to which a line must be drawn from the vertex making the greatest angle with the curve.

XIII. QUESTION 383, *by* PALABA.

Two cylinders of equal diameters, open at the top, are filled with water, and one of them is placed upon the other; a small orifice being made in the base of each, it is required to ascertain the time in which the lower cylinder will be completely emptied?

XIV. QUESTION 384, *by* PALABA.

Find the equation of the curve of which this is the property; if from a fixed point in the axis a perpendicular be drawn to it, and produced to meet a tangent to any point in the curve, the length of this perpendicular and tangent together shall be double the length of the curve between the vertex and the point from which the tangent was drawn.

XV. QUESTION 385, *by* PALABA.

If the sine of incidence : sine of refraction :: 1 : n, r and r' the radii of the surfaces, and t its thickness, the distance f of the principal focus from the focal centre may be *accurately* determined from this expression,

$$\frac{1}{f} = \frac{1-n}{n} \cdot \left\{ \frac{1}{r} + \frac{1}{r'} - \frac{1-n}{rr'} \cdot t \right\}.$$

Required the investigation?

XVI. QUESTION 386, *by* PALABA.

A prismatic vessel of given dimensions with its sides vertical is filled with water; there are two given and equal orifices, one at the bottom the other bisecting the altitude; required the time of emptying the upper half, supposing both orifices to be opened at the same instant?

XVII. QUESTION 387, *by* PALABA.

TB, BC are the subtangent and ordinate of a curve whose vertex is A, and the tangent of the angle TCA is to the tangent of the angle ACB in a given ratio. What is the nature of the curve?

XVIII. QUESTION 338, *by Mr.* T. S. EVANS.

The most expeditious method of determining the latitude appears to be, by observing a number of altitudes near the meridian, with a repeating circle; and the following simple formula reduces them with great facility to the meridian altitude;

$\sin \overset{\cdot}{\overline{\text{ZS}}} = \text{versin } \text{P} \times \frac{\sin \text{PS} \sin \text{PZ}}{\sin \text{ZS}}$. Required its investigation?

XIX. QUESTION 389, *by* G. V.

Let a be an arc of a circle of which the radius is unity, then

$$a^4 = 5 \times 48^2 \times \frac{3 + \cos a - 4 \cos \frac{1}{2}a}{237 - \cos a + 124 \cos \frac{1}{2}a}, \text{ nearly.}$$

Required the proof?

XX. PRIZE QUESTION 390, *by* M. G——E.

Three circles being given on the same plane; if two tangents be drawn from one of them to each of the other two, the line joining the intersections of the chords of contact will meet the first circle in two points which are the points of contact of this circle with two other circles, one of which touches the three given circles externally, and the other internally. Required the demonstration?

Solutions to these Questions must come to hand (post paid) by the first day of Aug. 1816.

THE

MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XVII.

I QUESTION 391, *by Mr.* John Baines, Junr.

To find two rational fractions, either of which being added to the square of the other shall make the same sum, and also that their difference shall be a biquadrate number.

II. QUESTION 392, *by the Rev. Mr.* L. Evans, *Royal Military Academy, Woolwich.*

The ordnance clock at this Institution, being placed in my observatory, for the intent of rating it, prior to its removal to Scotland, for an experimental purpose of ascertaining the true figure of the earth; I adjusted the length of its pendulum precisely to beat seconds, sidereal time. I would now request to know, how many seconds per day, will the clock gain or lose, in the latitude of 57° 5′ north, being that of Aberdeen, upon the hypothesis of Newton's consideration of the equatorial diameter being to the polar as 230 to 229.

III. QUESTION 393, *by* Italicus.

To describe three circles in a triangle, each of which may touch the other two, and also two sides of the triangle.

IV. QUESTION 394, *by* G. V.

Prove, that if from the extremities of the side of a pentagon inscribed in a circle, straight lines be drawn to the middle of the arc, subtended by the adjacent side, their difference is equal to the radius; the sum of their squares is three times the square of the radius; and the rectangle contained by them is equal to the square of the radius.

V QUESTION 395, *by* Math Pemb. Cantab.

An ellipse and a circle have the same centre, the axis major of the ellipse = $\frac{3}{2}$ × diameter of the circle: to determine the points of intersection of the two curves, supposing the foci to cut the distance between the extremities of the diameter and axis in a given ratio; and to determine that ratio when the ellipse and circle just touch each other.

VI. QUESTION 396, *by* G. V.

A horizontal dial constructed for a given latitude, is set up in a latitude a little different, what will be the error in the time it shews, at a given hour on a given day?

VII. QUESTION 397, *by* Gallicus.

Having given the position of any number of towns whatever, in the same plane, to connect them by a system of canals, of which, the total length shall be the least possible.

VIII. QUESTION 398, *by* Vicinus.

The extremities of a straight line on the ground are marked by two upright poles beyond which there is no access to any point in the line: shew how a person by planting poles in the ground (but without using any other instrument) may determine any number of points in the line between its extremities.

IX. QUESTION 399, *by* Gallicus.

Suppose that A a point in a line of the second order is given; a point P may be found such that if any chord BC be drawn through it, the lines AB, AC shall contain a right angle

X. QUESTION 400, *by* Peregrinator.

Let there be a system of four points, accessible (and either visible or invisible from one another). Let there be a system of four other points, inaccessible, but each of them visible from all the four points of the first system. The angles are observed under which the distances of all the four points of the second system are seen from each point of the first system: required the positions of these eight points in respect of one another?

XI. QUESTION 401, *by Mr.* W. Wallace, R. M. C.

A beam is supported by resting on a prop, and with one end on a curve surface; Find the nature of the curve, so that the beam may be in equilibrio in any position whatever. Also supposing the nature of the curve given, find the position of the beam when it is in equilibrio.

XII. QUESTION 402, *by Mr.* CUNLIFFE, R. M. C.

At the time of the equinox, in latitude 60°, suppose a person sets out at noon, on a clear day, and travels till sun set, in the direction of his shadow, at the uniform rate of 5 miles an hour; how far will he then be from the place from whence he set out, supposing the journey performed upon an horizontal plane: and what will be the area of the space included between the traveller's path and a right line connecting the two extremities thereof.

XIII. QUESTION 403, *by* AMICUS.

If from a point without a conic section two tangents be drawn to the curve, and also any other straight line to meet it in two points, the lines joining these points and one of the points of contact shall cut off equal segments from the chord which passes through the other point of contact parallel to the tangent.

XIV. QUESTION 404, *by Mr.* CUNLIFFE, R. M. C.

In a given hollow cone, it is required to place a solid, the axis of which shall coincide with the axis of the cone; and the property of the solid shall be such, that the area of any ellipse, formed by a plane touching the solid, and cutting the cone, shall be of the same given magnitude: Required the equation of the curve by whose revolution the solid will be generated.

XV QUESTION 405, *by Mr.* W. WALLACE, R. M. C.

Let a be an arc of a circle, and c its chord, also let c', c'', c''', c^{IV}, &c. be the chords of its half, its fourth, its eighth, its sixteenth, &c. Prove that

$\sin a = c - c\,c'\,c'' + c\,c'\,c''\,c'''\,c^{IV} - c\,c'\,c''\,c'''\,c^{IV}\,c^{V}\,c^{VI} + \&c.$

$\cos a = 1 - c\,c' + c\,c'\,c''\,c''' - c\,c'\,c''\,c'''\,c^{IV}\,c^{V} + \&c.$

both series being continued *ad infinitum*.

XVI. QUESTION 406, *by Mr.* HERSCHEL.

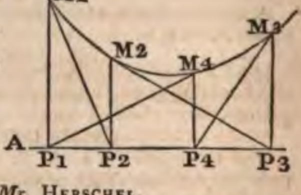
AC, CB are the major and minor semiaxes of a conic section, Cp any semidiameter. Draw APO parallel to Cp meeting CB, produced if necessary, in O. It is required to prove that OA, AP $= 2\,Cp^2$.

XVII. QUESTION 407, *by Mr.* HERSCHEL.

The wind blows a spider's web hanging freely upon two points (and supposed destitute of relative gravity) into a curvilinear form. Required the nature of the curve and the law of tension.

XVIII. QUESTION 408, *by Mr.* HERSCHEL.

To determine the nature of a curve such, that drawing any ordinate $P_1 M_1$ and tangent $M_1 P_2$, again erecting a second ordinate $P_2 M_2$ and drawing a second tangent $M_2 P_3$, if this be repeated n times, the last tangent ($M_4 P_1$ in the figure,) shall invariably meet the axis at the foot of the first ordinate.

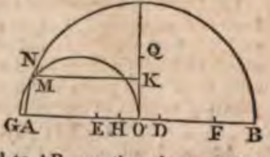


XIX. QUESTION 409, *by Mr.* HERSCHEL.

To determine the nature of a curve such, that any series of tangents bein drawn as in the preceding question, the subtangents FP_1, $P_1 P_2$, $P_2 P_3$ &c. in the same series shall be all equal, however they may differ in different series originating from a different position of PM.

XX. PRIZE QUESTION 410, *by Mr.* LOWRY, R. M. C.

Required a geometrical demonstration of the following method of constructing a regular polygon of seventeen sides in a circle: Draw the radius CO at right angles to the diameter AB; on OC and OB, take OQ equal to the half, and OD equal to an eighth part of the radius; make DE and DF each equal to DQ, and EC and FH respectively equal to EQ and FQ; take OK a mean proportional between OH and OQ, and through K, draw KM parallel to AB, meeting the semicircle described on OG in M; draw MN parallel to OC cutting the given circle in N, the arc AN is the seventeenth part of the whole circumference.



Solutions to these questions must come to hand (post paid) by the first day of June, 1819.

THE

MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XVIII.

Vol 5

I. QUESTION 411, *by* Theodosius.

Having given the three sides of a spherical triangle, it is required to find, in terms of these sides, the radii of the inscribed and circumscribing circles and the distance of their centres.

II. QUESTION 412, *by* Geometricus.

In a given polygon to inscribe another of the same number of sides, having its angles respectively equal to given angles, and either its perimeter or area a given magnitude.

III. QUESTION 413 *by* Analyticus.

Two vessels capable of containing a and b gallons are filled with mixtures of wine and water, in given proportions for each vessel; two equal measures, c gallons each, are filled at the same time, namely one from each vessel, and then emptied into the other. Supposing this operation to be repeated n times, what would then be the proportion of wine and water in each vessel?

IV. QUESTION 414, *by* Gallicus.

Find the position of a plane so that if a given triangle be projected orthographically upon it, the projection may be a triangle similar to a given triangle.

V. QUESTION 415, *by* Mechanicus.

The chord of an arc described by the lower extremity of a simple pendulum being given in length, it is required to determine the length of the pendulum, when its vibrations through this arc are performed in the least time.

VI. QUESTION 416, *by* Mechanicus.

If the point of suspension of a simple pendulum, instead of being fixed, be constrained to move along a straight horizontal line with a given uniform velocity, it is required to assign the nature of the curve described by its lower extremity, and determine the other circumstances of its motion.

VII. QUESTION 417, *by* Astronomicus.

Knowing the declination of a star, and the time of its rising or setting for any determinate place on the globe, it is required to determine the correction which must be made to give the time of rising or setting for any other place near to the former.

VIII. QUESTION 418, *by* Edinburgensis.

An obelisk of a given height stands on an inclined plane given by position: What is the *locus* of that point on the plane at which a statue of a given height on the top of the obelisk subtends a given angle?

IX. QUESTION 419, *by* Palaba.

The distance of the centre of gravity of the surface of a certain solid from the vertex is equal to half the abscissa: determine the nature of the curve by the revolution of which round its axis the surface was generated.

X. QUESTION 420, *by* Proteus.

It is required to inscribe a triangle in a given circle, such, that the ratio of two of its sides shall be given, and the rectangle of the segments of the other side, made by a perpendicular, from the opposite angle shall be the greatest possible.

XI. QUESTION 421, *by* Scandinianus.

Suppose two straight lines and a point without them to be given by position; two points, given by position, may be found, such, that if any two parallels be drawn through them, to meet the lines given by position, one of the diagonals of the trapezium thus formed shall pass through the other given point.

XII. QUESTION 422, *by* LAPUTIENSIS.

It is required to draw two right lines, from the focus of a given conic section, to terminate in the curve, so that these lines shall have a given ratio to each other, and shall also include a given angle.

XIII. QUESTION 423, *by* C. B.

In the circle of the nth order whose equation is $x^{2n} + y^{2n} = a^{2n}$, any point C being taken, and the abscissa and ordinate OF and CF being drawn, if the triangle OCF is placed in the situation OEB, then the point B shall be situated in the curve, and the sum of the two arcs DB and DC shall always be constant.

XIV. QUESTION 424, *by Mr.* CUNLIFFE, R. M. C.

Given the segments of the base, made by a perpendicular from the vertical angle, and the rectangle of the other two sides, to construct the plane triangle by a method purely geometrical.

XV QUESTION 425, *by Mr.* CUNLIFFE, R. M. C.

If circles are described upon the sides of a plane triangle as diameters; and perpendiculars drawn from the angles to the sides; the segments of these perpendiculars, intercepted by each side, and the circumference of its respective circle, will be equal to each other: required the demonstration?

XVI. QUESTION 426, *by* LAPUTIENSIS.

Suppose two given weights, appended at given points, to a flexible line of a given length, considered without weight and fastened at the ends to two given points in the same horizontal line, and imagine a third given weight, appended to the line, between the other two, but at liberty to slide freely thereon by means of a ring; it is required to determine the position of the ring, and the points in the line, to which the other two weights are appended, when at rest, in the above mentioned circumstances.

XVII. QUESTION 427, *by* C. B.

Prove that $\frac{1}{2} = \frac{1}{\cos\theta} - \frac{1}{\cos 2\theta} + \frac{1}{\cos 3\theta} - \&c.$

XVIII. QUESTION 428, *by* C. B.

Prove that the sum of the series

$$\frac{1}{1.3.....2n+1} - \frac{1}{3.5....2n+3} + \&c. \text{ is equal to}$$

$$\frac{2^{-1}}{1.2.....n}\left\{\frac{\pi}{2} - \frac{1}{1} - \frac{1}{1.3} - \frac{1.2}{1.3.5} - \&c. \text{ to } n \text{ terms}\right\}.$$

XIX. QUESTION 429, *by* C. B.

Prove that the series

$$\frac{1}{1.2.3...n} - \frac{1}{2.3.4....n+1} \mp \frac{1}{3.4.....n+2} - \&c. \text{ is equal to}$$

$$\frac{1}{1.2...(n-1)}\left\{2^{n-1}\log. 2 - \frac{2^{n-2}}{1} - \frac{2^{n-3}}{2} - \&c - \frac{2^{0}}{n-1}\right\};$$

XX. PRIZE QUESTION 430, *by* MECHANICUS.

If two bodies, connected by an inflexible and inextensible line, be placed on an horizontal plane, and one of them be constrained to move in a straight line given by position, with a uniform velocity: it is required to determine the circumstances of the motion of the other body, and the nature of the curve it describes.

* * *

Solutions to these questions must come to hand (post paid) by the first day of December, 1819.

THE
MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

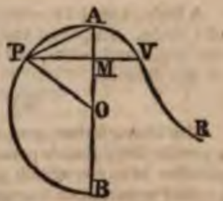
To be answered in Number XIX. Vol 5.

I. QUESTION 431, *by* PALABA.

A and B travelled on the same road and at the same rate from H to L. At the 50th mile stone from L, A overtook a drove of geese, which were proceeding at the rate of three miles in two hours; and two hours afterwards met a stage waggon, which was moving at the rate of nine miles in four hours. B overtook the same drove of geese at the 45th mile stone and met the same stage waggon exactly forty minutes before he came to the 31st mile stone. Where was B when A reached L?

II. QUESTION 432, *by* PALABA.

The curve AVR and semicircle APB have the same abscissa, the ordinate MV = tan ½ arc AP: prove that the area AMV = twice the segment AP.



III. QUESTION 433, *by* PALABA.

A parabola revolves round its axis, which is vertical, in a given time, and the angular motion will just prevent a body at any point of the curve from descending: required the parameter of the parabola?

IV. QUESTION 434, *by* PALABA.

To divide a given arc (A), less then a quadrant, into two such parts (p) and (q) that $\tan^m p \times \tan^n q$ may = max.

V. QUESTION 435, *by* G. V.

A normal at any point of an equilateral hyperbola is equal to the distance of that point from the centre. Required the demonstration.

VI. QUESTION 436, *by* G. V.

C is the centre of an ellipse and F either focus, PH a tangent at P: draw the diameter PCp and pF meeting PH in H. Prove that pH = the transverse diameter.

VII. QUESTION 437, *by* G. V.

P is any point in the diameter of a circle and PB a perpendicular to the diameter: draw any chord PA, and a tangent AB meeting PB in B, and draw BD and CE (C the centre) perpendicular to PA; then PE = DA. Required a demonstration.

VIII. QUESTION 438, *by* PROTEUS.

Within a given triangle suppose another triangle to be inscribed, by joining the middle points of its sides; and again, within this triangle, suppose another triangle to be inscribed, by joining the middle points of its sides, and so on ad infinitum: What will be the limit of the aggregate, of the sum of the squares of the sides of all the triangles so formed?

IX. QUESTION 439, *by Mr.* CUNLIFFE.

Let any right line be drawn through the focus of a given conic section, terminating in the curve; then a fourth proportional to the whole line and the two segments thereof, made by the focus, will always be of the same constant length. Required a demonstration.

X. QUESTION 440, *by* ———.

A triangle being given, it is required to describe three circles, so that each circle shall touch the other two and a side of the triangle at the point of bisection.

XI. QUESTION 441, *by* ———.

Required the curve that has at each point the radius of curvature a fourth proportional to the abscissa, the ordinate, and a given straight line.

XII. QUESTION 442, *by* PALABA.

To determine the nature of the curve such that the perpendicular from a given point upon the tangent shall be a mean proportional between a given line and the segement of the axis intercepted between the tangent and this same given point.

XIII. QUESTION 443, *by* PALABA.

To determine the equation to the curve whose tangent is a mean proportional between the segment of the axis intercepted between it and a given point, and that same segment augmented by a given line.

XIV. QUESTION 444, *by* PALABA.

A body, urged by a force perpendicular to the horizon, describes the quadrant of a circle. Required the law of force which will make it recede uniformly from the horizontal radius, and the time elapsed and the velocity acquired at any point of the descent.

XV. QUESTION 445, *by* ———.

The characteristic property of the circle is, that all the chords which pass through a certain determinate point in its plane are equal: but there exists an indefinite number of curves which possess this property. It is required to find the most general equation to curves of this nature.

XVI. QUESTION 446, *by Mr.* CUNLIFFE.

A given rod or beam, has one end suspended by a cord of a given length, fixed at a given point above an inclined plane of a given inclination; it is required to determine the position of the beam, weight sustained by the chord, and pressure against the inclined plane, when the beam is in equilibrio.

XVII. QUESTION 447, *by* G. V.

Find the rectification and quadrature of the magnetic curve, which is such, that if lines be drawn from the poles to any point in the curve and φ and ψ be the angles they make with the line joining them, $\cos\varphi + \cos\psi = c$, a constant quantity.

XVIII. QUESTION 448, *by* ERATOSTHENES.

Required a number consisting of six digits $abcdef$ such that, its multiples by 1, 2, 3, 4, 5, 6 may contain amongst them the following arrangements:

$$\begin{array}{cccccc} a & b & c & d & e & f \\ b & c & d & e & f & a \\ c & d & e & f & a & b \\ d & e & f & a & b & c \\ e & f & a & b & c & d \\ f & a & b & c & d & e. \end{array}$$

XIX. QUESTION 449, *by* ERATOSTHENES.

It is well-known that the formula $x^2 + x + 41$ contains a great number of primes: prove that when x is of either of the two following forms it cannot be a prime, $x = 53a + 8$ and $x = 97a - 8$.

XX. PRIZE QUESTION 450, *by* MECHANICUS.

If the beam in question 401 instead of resting upon the curve, be at liberty to slide freely over the prop and descend by the force of gravity, it is required to determine the motion of its centre of gravity.

Solutions to these questions must come to hand (post paid) by the first day of June, 1820.

THE

MATHEMATICAL REPOSITORY,

VOL. IV. PART I.

MATHEMATICAL QUESTIONS.

ARTICLE I.

Solutions to Questions proposed in Number XII.

I. QUESTION 331, *by Mr.* JOHN HYNES, *Dublin.*

To find any number of squares whose sum and product are equal.

SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

In order to make the solution quite easy, I shall begin with finding two squares whose sum and product shall be equal to each other; and afterwards proceed gradually to finding three, four, and five squares, whose sum and continual product shall be equal to each other, till the way of extending the solution to any number of squares, is sufficiently clear and evident.

1. To find two square numbers whose sum and product shall be equal to each other.

Let x^2 and y^2 denote the two squares; then, by the question, $x^2y^2 = x^2 + y^2$, whence $x^2 = \frac{y^2}{y^2 - 1}\Big| =$ a square; therefore $y^2 - 1$ must be a square. Now $y^2 - 1$ will evidently be a square when $y = (m^2 + n^2) \div 2mn$ where m and n may be taken at pleasure. For then $x^2 = \frac{y^2}{y^2 - 1} = \frac{(m^2 + n^2)^2}{(m^2 - n^2)^2}$, $y^2 = \frac{(m^2 + n^2)^2}{4m^2n^2}$.

Take $m = 2$ and $n = 1$, then $x^2 = \frac{25}{9}$ and $y^2 = \frac{25}{16}$.

2. To find three rational squares whose sum shall be equal to their continual product.

Let the squares be denoted by x^2, y^2 and z^2: Then, by the question, $x^2y^2z^2 = x^2 + y^2 + z^2$, hence $x^2 = \frac{y^2 + z^2}{y^2z^2 - 1}$: whence it appears that $y^2 + z^2$ and $y^2z^2 - 1$ must be both squares. Put $y^2 + z^2 = (y + z - n)^2$, whence $z = \frac{2ny - n^2}{2(y - n)}$, and hence $y^2z^2 - 1 = \frac{y^2(2ny - n^2)^2}{4(y - n)^2} - 1 = \frac{y^2(2ny - n^2)^2 - 4(y - n)^2}{4(y - n)^2}$ $=$ a square; therefore $y^2(2ny - n^2)^2 - 4(y - n)^2 = 4n^2y^4 - 4n^3y^3 + n^4y^2 - 4y^2 + 8ny - 4n^2 =$ a square. Assume $2ny^2 - n^2y - \frac{1}{n}$ for its root; that is, put $4n^2y^4 - 4n^3y^3 + n^4y^2 - 4y^2 + 8ny - 4n^2 = \left(2ny^2 - n^2y - \frac{1}{n}\right)^2 = 4n^2y^4 - 4n^3y^3 + n^4y^2 - 4y^2 + 2ny + \frac{1}{n^2}$: whence $6ny = 4n^2 + \frac{1}{n^2} = \frac{4n^4 + 1}{n^2}$ and $y = \frac{4n^4 + 1}{6n^3}$ where n may be taken at pleasure.

Take $n = 1$, then $y = \frac{5}{6}$, $z = \frac{2ny - n^2}{2(y - n)} = -2$, $x^2 = \frac{y^2 + z^2}{y^2z^2 - 1} = \frac{169}{64}$: Therefore the three squares are $\frac{169}{64}$, $\frac{25}{36}$ and 4.

Take $n = 2$, then $y = \frac{65}{48}$, $z = \frac{2ny - n^2}{2(y - n)} = -\frac{34}{31}$, $x^2 = \frac{(2593)^2}{(1634)^2}$, and therefore the three squares in this case are $\frac{(2593)^2}{(1634)^2}$, $\frac{(65)^2}{(48)^2}$ and $\frac{(34)^2}{(31)^2}$.

3. To find four squares, such, that their sum and continual product may be equal to each other.

Let v^2, x^2, y^2 and z^2 denote the four squares: Then, by the question, $v^2x^2y^2z^2 = v^2 + x^2 + y^2 + z^2$; hence $v^2 = (x^2 + y^2 + z^2) \div x^2y^2z^2 - 1$. The last expression will manifestly be a square when $x^2 + y^2 + z^2$ and $x^2y^2z^2 - 1$ are both squares. Put $x^2 + y^2 + z^2 = (x + y - z)^2 = x^2 + 2xy + y^2 - 2z(x + y) + z^2$; whence $z = xy \div (x + y)$, and $x^2y^2z^2 - 1$

$= x^4y^4 \div (x+y)^2 - 1 = (x^4y^4 - (x+y)^2) \div (x+y)^2 =$ a square; therefore $x^4y^4 - (x+y)^2 = x^4y^4 - x^2 - 2xy - y^2$ $=$ a square: Assume $x^2y^2 - \frac{1}{2y^2}$ for the root, that is, put $x^4y^4 - x^2 - 2xy - y^2 = \left(x^2y^2 - \frac{1}{2y^2}\right)^2 = x^4y^4 - x^2 + \frac{1}{4y^4}$; whence $x = -\frac{4y^6+1}{8y^5}$, where y may be taken at pleasure.

Take $y = 1$, then $x = -\frac{5}{8}$, $z = -\frac{5}{3}$ and $v^2 = 49$; and therefore 49, $\frac{25}{64}$, 1, and $\frac{25}{9}$ are four squares that will answer.

Or thus:

Since $x^2 + y^2 + z^2$ and $x^2y^2z^2 - 1$ are both to be squares: put $x^2 + y^2 + z^2 = (x + y - m)^2 = x^2 + 2xy + y^2 - 2m(x + y) + m^2$: then $x = \frac{2my + z^2 - m^2}{2(y-m)}$, and $x^2y^2z^2 - 1 = \frac{y^2z^2(2my + z^2 - m^2)^2}{4(y-m)^2} - 1 = \frac{y^2z^2(2my + z^2 - m^2)^2 - 4(y-m)^2}{4(y-m)^2}$ $=$ a square: therefore $y^2z^2(2my + z^2 - m^2)^2 - 4(y-m)^2$ $= 4m^2z^2y^4 + 4mz^2y^3(z^2 - m^2) + y^2z^2(z^2 - m^2)^2 - 4y^2 + 8my - 4m^2 =$ a square. Assume $2mzy^2 + zy(z^2 - m^2) - g$ for its root; that is, put $4m^2z^2y^4 + 4mz^2y^3(z^2 - m^2) + y^2z^2(z^2 - m^2)^2 - 4y^2 + 8my - 4m^2 = (2mzy^2 + zy(z^2 - m^2) - g)^2$ $= 4m^2z^2y^4 + 4mz^2y^3(z^2 - m^2) + z^2y^2(z^2 - m^2)^2 - 4gmzy^2$ $- 2gzy(z^2 - m^2) + g^2$. Put $4gmz = 4$, or $g = \frac{1}{mz}$, in order to take away all the terms in which any power of y above the first is concerned; and from the equality of the remaining terms, viz. $8my - 4m^2 = -2gzy(z^2 - m^2) + g^2$, $y = \frac{4m^2 + g^2}{8m + 2gz(z^2 - m^2)} = \frac{4m^2 + 1 \div m^2z^2}{8m + 2(z^2 - m^2) \div m} = \frac{4m^4z^2 + 1}{6m^3z^2 + 2mz^4} = \frac{4m^4z^2 + 1}{2mz^2(3m^2 + z^2)}$, where m and z may be taken at pleasure.

Take z and m each $= 1$, then $y = \frac{5}{8}$, $x = -\frac{5}{3}$, and $v^2 = 49$; therefore 49, 1, $\frac{25}{64}$ and $\frac{25}{9}$ are four squares that will answer.

Take $z = 1$ and $m = \frac{1}{2}$, then $y = \frac{5}{7}$, $x = \frac{41}{12}$ and $v^2 =$

$\frac{(305)^2}{(187)^2}$; therefore 1, $\frac{5^2}{7^2}$, $\frac{(41)^2}{(12)^2}$ and $\frac{(305)^2}{(187)^2}$, are four squares that will answer.

4. To find five squares whose sum and continual product shall be equal to each other.

Let the squares be denoted by u^2, v^2, x^2, y^2, and z^2. By the question $u^2v^2x^2y^2z^2 = u^2 + v^2 + x^2 + y^2 + z^2$: whence $u^2 = \frac{v^2 + x^2 + y^2 + z^2}{v^2x^2y^2z^2 - 1}$: put $v^2 + x^2 + y^2 + z^2 = (x + y - m)^2 = x^2 + 2xy + y^2 - 2m(x + y) + m^2$: whence $x = \frac{2my + v^2 + z^2 - m^2}{2(y - m)}$, and hence $v^2x^2y^2z^2 - 1 =$ $\frac{v^2y^2z^2(2my + v^2 + z^2 - m^2)^2}{4(y - m)^2} - 1 =$ $\frac{v^2y^2z^2(2my + v^2 + z^2 - m^2)^2 - 4(y - m)^2}{4(y - m)^2} =$ a square, and therefore $v^2y^2z^2(2my + v^2 + z^2 - m^2)^2 - 4(y - m)^2 = 4m^2v^2z^2y^4 + 4mv^2z^2y^3(v^2 + z^2 - m^2) + v^2z^2y^2(v^2 + z^2 - m^2)^2 - 4y^2 + 8my - 4m^2 =$ a square $= (2mvzy^2 + vzy(v^2 + z^2 - m^2) - g)^2 = 4m^2v^2z^2y^4 + 4mv^2z^2y^3(v^2 + z^2 - m^2) + v^2z^2y^2(v^2 + z^2 - m^2)^2 - 4gmvzy^2 - 2gvzy(v^2 + z^2 - m^2) + g^2$. Put $4gmvz = 4$, or $g = \frac{1}{mvz}$, in order to take away all the powers of y above the first; and the remaining terms of the expression will be $8my - 4m^2 = -2gvzy(v^2 + z^2 - m^2) + g^2$; whence $y = \frac{4m^2 + g^2}{8m + 2gvz(v^2 + z^2 - m^2)} =$ $\frac{4m^2 + 1 \div m^2v^2z^2}{8m + (2 \div m)(v^2 + z^2 - m^2)} = \frac{4m^4v^2z^2 + 1}{8m^3v^2z^2 + 2mv^2z^2(v^2 + z^2 - m^2)} =$ $\frac{4m^4v^2z^2 + 1}{6m^3v^2z^2 + 2mv^2z^2(v^2 + z^2)} = \frac{4m^4v^2z^2 + 1}{2mv^2z^2(3m^2 + v^2 + z^2)}$, where m, v, and z may be taken at pleasure.

Take $m = 1$, $v = 2$, and $z = \frac{1}{2}$, then $y = \frac{10}{29}$, and $x = -\frac{457}{152}$; and hence $u^2 = \frac{(16141)^2}{(1206)^2}$. Therefore 4, $\frac{1}{4}$, $\frac{(10)^2}{(29)^2}$, $\frac{(457)^2}{(152)^2}$, and $\frac{(16141)^2}{(1206)^2}$ are five squares that will answer.

If the process and results of the two preceding cases are attentively considered, there will be no difficulty in extending the solution to as many squares as we please, without any further calculation. For let w^2, u^2, v^2, x^2, y^2 and z^2 denote six squares whose continual product is equal to their sum: then we shall have $y = \frac{4m^4u^2v^2z^2 + 1}{2mu^2v^2z^2(3m^2 + u^2 + v^2 + z^2)}$, $x = \frac{2my + u^2 + v^2 + z^2 - m^2}{2(y - m)}$ and $w^2 = \frac{u^2 + v^2 + x^2 + y^2 + z^2}{u^2v^2x^2y^2z^2 - 1}$.

It also appears from what has been done, that all the squares excepting three, may be assumed at pleasure: and the roots of the three other squares are to be found by means of the preceding formulæ.

II. QUESTION 332, *by Mr.* JOHN HYNES.

To find two fractions, such, that the sum and sum of their squares shall both be rational squares; and either of them being added to the square of the other shall make the same square.

SOLUTION, *by Mr.* CUNLIFFE.

Let the two fractions be denoted by $\frac{x}{x+y}$ and $\frac{y}{x+y}$, for the sum of these is obviously 1, and is therefore a square number. The sum of their squares is $\frac{x^2+y^2}{(x+y)^2}$, which will evidently be a square when x^2+y^2 is a square. Again either of the fractions being added to the square of the other is $\frac{x^2+xy+y^2}{(x+y)^2}$ which will also evidently be a square when x^2+xy+y^2 is a square.

The question is therefore reduced to finding such rational values of x and y as will make x^2+y^2 and x^2+xy+y^2 both squares.

Put $x = m^2 - n^2$ and $y = 2mn$, then $x^2 + y^2 = (m^2 - n^2)^2 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2 =$ a square: Also, $x^2 + xy + y^2 = m^4 + 2m^3n + 2m^2n^2 - 2mn^3 + n^4$ is to be a square. Assume $m^2 + mn + n^2$ for the root of the preceding square; then $m^4 + 2m^3n + 2m^2n^2 - 2mn^3 + n^4 = (m^2 + mn + n^2)^2 = m^4 + 2m^3n + 3m^2n^2 + 2mn^3 + n^4$: whence $\frac{m}{n} = \frac{4}{-1}$. Wherefore we may take $m = 4$ and $n = -1$; but these values of m and n would manifestly give the value of y negative. In order therefore to obtain a positive value of y, take $m = 4$ and $n = s - 1$; then $x = m^2 - n^2 = 15 + 2s - s^2$, and $y = 2mn = 8(s - 1)$, and from hence $x^2 + xy + y^2 = 169 + 36s + 62s^2 - 12s^3 + s^4$ which is to be a square. Assume $13 + 6s -$

s^2 for its root, that is, put $169 + 36s + 62s^2 - 12s^3 + s^4 = (13 + 6s - s^2)^2 = 169 + 156s + 10s^2 - 12s^3 + s^4$; whence $s = \frac{30}{13}$, $n = s - 1 = \frac{17}{13}$, $x = m^2 - n^2 = \frac{2415}{169}$, $y = 2mn = \frac{1768}{169}$, and rejecting the common denominator, we may take $x = 2415$, and $y = 1768$, and hence the required fractions will be $\frac{x}{x+y} = \frac{2415}{4183}$ and $\frac{y}{x+y} = \frac{1768}{4183}$.

III. QUESTION 333, *by* JUNIUS.

Required the general value of x in the equation $x^2 - 23y^2 = 1$?

SOLUTION, *by Mr.* CUNLIFFE.

The given equation by transposition becomes $1 + 23y^2 = x^2$. When the coefficient of y^2 is within two of being a square number, the integer value of y, which will make $1 + 23y^2$ a square, may be readily found, as appears from what follows.

Put $q^2 - 2$ for the coefficient of y^2 in the given equation, that is, let the given equation be $1 + y^2(q^2 - 2) = x^2$. Assume $x = qy - 1$, then $1 + y^2(q^2 - 2) = (qy - 1)^2 = q^2y^2 - 2qy + 1$, whence $y = q$, and $x = qy - 1 = q^2 - 1$: and comparing the foregoing expression with the proposed example, we have $q^2 - 2 = 23$, $q^2 = 25$, $q = 5$; therefore $y = 5$, and $x = q^2 - 1 = 24$.

At page 383, art. 180, Barlow's Theory of Numbers, we have the following general formulæ, expressing the values of x and y, viz.

$$x = \frac{(p + q\sqrt{23})^m + (p - q\sqrt{23})^m}{2}, \text{ and}$$

$$y = \frac{(p + q\sqrt{23})^m - (p - q\sqrt{23})^m}{2\sqrt{23}};$$

in which p and q denote particular values of x and y that will answer, and m is any integer number.

In the present case $p = 24$, $q = 5$, and take $m = 2$; then $x = 1151$ and $y = 240$.

But the most general and comprehensive method of proceeding for finding the values of p and q, is by putting the square root of the coefficient of y^2 into a continued fraction, and from thence forming a series of converging fractions, the terms of one of which will be the values of p and q. The method of doing this is very fully explained and exemplified in Mr. Barlow's Theory of Numbers.

IV. QUESTION [illegible] ALLOW.

There are two such quantities that the sum of their squares exceeds their sum by a; and that the sum of their fourth powers together with their sum exceeds twice the sum of their cubes by b; it is required to find them without resolving any equation higher than a quadratic?

FIRST SOLUTION, *by Mr.* CALLOW, *the Proposer.*

Put x and y for the required quantities; then, by the question, $x^2 + y^2 - (x + y) = a$, and $x^4 + y^4 + (x + y) - 2(x^3 + y^3) = b$; adding these two equations, we have

$$x^4 + y^4 - 2(x^3 + y^3) + x^2 + y^2 = b + a.$$

Put now $x^2 - x = m$, and $y^2 - y = n$, then our first and last equations become $m + n = a$, and $m^2 + n^2 = b + a$; m and n are therefore the two roots of the quadratic, $n^2 - an + \frac{1}{2}(a^2 - b - a) = 0$, and since $x^2 - x = m$ and $y^2 - y = n$, x and y are also known.

For an example, let $a = 8$, and $b = 32$, then the quadratic for determining m and n becomes $n^2 - 8n + 12 = 0$, of which the two roots are 2 and 6: take $m = 2$, and $n = 6$, then our quadratics for determining x and y become $x^2 - x = 2$ and $y^2 - y = 6$: whence $x = +2$ or -1, and $y = +3$ or -2: therefore 2, and 3; or -1, and -2; or $+2$, and -2; or $+3$, and -1, are four different sets of numbers which answer the conditions of the question.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Let z denote half the sum and v half the difference of the quantities; then $z + v$ will denote the greater, and $z - v$ the less; $2(z^2 + v^2)$ the sum of the squares, $2z^3 + 6zv$ the sum of the cubes, and $2z^4 + 12z^2v^2 + 2v^4$ the sum of the biquadrates or fourth powers. From whence and by the question,

$$z^2 + v^2 - z = \tfrac{1}{2}a, \text{ and}$$
$$z^4 + 6z^2v^2 + v^4 + z - 2z^3 - 6zv^2 = \tfrac{1}{2}b.$$

The latter equation being subtracted from the square of the former, leaves

$$4zv^2 - 4z^2v^2 + z^2 - z = \tfrac{1}{4}(a^2 - 2b):$$

but, from the first equation, $v^2 = \frac{1}{2}a + z - z^2$, by means of which exterminating v^2 from the preceding equation it becomes

$$(4z - 4z^2) \times (\tfrac{1}{2}a + z - z^2) + z^2 - z = \tfrac{1}{4}(a^2 - 2b),$$

which expression, when expanded is $4z^4 - 8z^3 - z^2 \times (2a - 5) + z \times (2a - 1) = \frac{1}{4}(a^2 - 2b)$; dividing by 4, $z^4 - 2z^3 - \frac{1}{4}(2a - 5)z^2 + \frac{1}{4}(2a - 1)z = \frac{1}{16}(a^2 - 2b)$; completing the square by adding $\frac{1}{64}(2a - 1)^2$ to each side, $z^4 - 2z^3 - \frac{1}{4}(2a - 5)z^2 + \frac{1}{4}(2a - 1)z + \frac{1}{64}(2a - 1)^2 = \frac{1}{64}(8a^2 - 4a + 1 - 8b)$, taking the square roots

$$z^2 - z - \tfrac{1}{8}(2a - 1) = \tfrac{1}{8}\surd(8a^2 - 4a + 1 - 8b),$$
$$z^2 - z = \tfrac{1}{8}(2a - 1) + \tfrac{1}{8}\surd(8a^2 - 4a + 1 - 8b).$$

From whence $z = \frac{1}{2} \pm \frac{1}{2}\surd(\frac{1}{2}(2a + 1) + \frac{1}{2}\surd(8a^2 - 4a + 1 - 8b))$.

Again, from the equation, $z^2 + v^2 - z = \frac{1}{2}a$,

$v^2 = \frac{1}{2}a - (z^2 - z) = \frac{1}{2}a - \frac{1}{8}(2a - 1) - \frac{1}{8}\surd(8a^2 - 4a + 1 - 8b)$

$= \frac{1}{8}(2a + 1) - \frac{1}{8}\surd(8a^2 - 4a + 1 - 8b)$; whence

$v = \frac{1}{2}\surd(\frac{1}{2}(2a + 1) - \frac{1}{2}\surd(8a^2 - 4a + 1 - 8b))$.

V. QUESTION 335, *by Mr.* CALLOW.

Having given $y^4 + qy^2 + ry + s = 0$, and $y - (x + m) = 0$: It is required to assign such a value to m, as will enable us to determine x and y, and thus to resolve a general biquadratic, supposing the solution of a general cubic to be known?

FIRST SOLUTION, *by Mr.* CALLOW, *the Proposer.*

From the second equation we get $y = x + m$, which substituted in the first, it becomes $x^4 + (4x^3 + 4xm^2 + 2qx + r)m + (6x^2 + q + m^2)m^2 + qx^2 + rx + s = 0$: feign $4x^3 + 4xm^2 + 2qx + r = 0$, whence $m^2 = -\left(x^2 + \frac{1}{2}q + \frac{\frac{1}{4}r}{x}\right)$, and our last equation therefore becomes (by substituting this value for m^2), after reductions, $x^6 + \frac{1}{2}qx^4 + (\frac{1}{16}q^2 - \frac{1}{4}s)x^2 - \frac{1}{16}r^2 = 0$: from which x is known by resolving this equation as a cubic, then x and m are also known, as also y the root of the general biquadratic: That is y, the root of the general biquadratic $y^4 + qy^2 + ry + s = 0$, is $= (x + m =) x \pm \surd -\left(x^2 + \frac{1}{2}q + \frac{\frac{1}{4}r}{x}\right)$, where x^2 is a root of $x^6 + \frac{1}{2}qx^4 + (\frac{1}{16}q^2 - \frac{1}{4}s)x^2 - \frac{1}{64}r^2 = 0$.

For an example, let $y^4 - 25y^2 + 60y - 36 = 0$, be proposed: here $q = -\frac{25}{2}$, $r = 60$, and $s = -36$, therefore our equation for finding x^2 becomes $x^6 - \frac{25}{2}x^4 + \frac{769}{16}x^2 - \frac{3600}{64} = 0$, whence $x^2 = 4$, or $\frac{9}{4}$, or $\frac{25}{4}$; take $x^2 = 4$, then

$x^2 = +2$, or -2, and $y = 2 \pm \sqrt{-(4 - \frac{25}{2} + \frac{15}{2})}$ $= 2 \pm 1$, or $y = -2 \pm \sqrt{-(4 - \frac{25}{2} + \frac{15}{2})} = -2 \pm 4$. Therefore $+1$, $+3$, $+2$, and -6, are the four required roots.

SECOND SOLUTION, *by* W.

It is here proposed to determine the value of y from the two equations

$$y^4 + qy^2 + ry + s = 0$$
$$y - x - m = 0.$$

Insert the latter value of y in the equation and divide by x^2, the result will be

$$x^2 + \frac{m^4 + qm^2 + rm + s}{x^2} + 4m \times \left(x + \frac{4m^3 + 2qm + r}{4mx}\right) + 6m^2 + q = 0.$$

Now as m is an undetermined quantity we are at liberty to assume $\left(\frac{4m^3 + 2qm + r}{4m}\right)^2 = m^4 + qm^2 + rm + s$, which by reduction gives the following cubic equation, from which the value of m may be calculated,

$$8rm^3 + (16s - 4q^2)m^2 - 4qrm - r^2 = 0.$$

Then if we assume z to represent $x + \frac{4m^3 + 2qm + r}{4mx}$, we shall find $x^2 + \frac{m^4 + qm^2 + rm + s}{x^2} = z^2 - \frac{4m^3 + 2qm + r}{2m}$, and $z^2 + 4mz = \frac{r - 8m^3}{2m}$, and from the value of z we deduce that of x by the resolution of the quadratic

$$x^2 - zx + \frac{4m^3 + 2qm + r}{4m} = 0.$$

This method of resolving the biquadratic is applicable to the complete equation $y^4 + py^3 + qy^2 + ry + s = 0$. It occurred to me more than twelve years ago; and, I think, speaking from recollection, the same or nearly similar results will be found in Dr. Waring's *Meditationes Algebraicæ*, although the Dr. derives them from a process altogether different.

VI. QUESTION 336, *by* AIEYOYO.

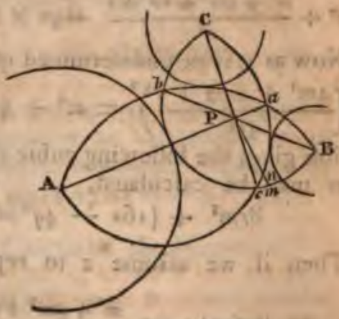
Demonstrate the following theorems.

1. If three circles be situated any how in a plane, and through the centres of every two a circle be described to touch the remaining one; the lines joining the centre of each of the circles with

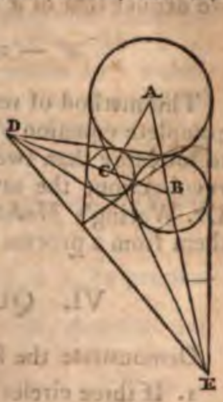
the point in which the circles passing through it meet each other, intersect in the same point.

2. If three circles touch one another, the one of which is a mean proportional between the other two, and if tangents be drawn to every two of these meeting their diameters externally; then it is known that the three points of section are in the same straight line; and it is required to prove that the square of this line has a given ratio to $M^2 + N^2 + MN$; M and N representing the radii of the first and second terms of the proportion.

SOLUTION, *by* ——.



1. Let the points A, B, C be the centres of the given circles, and *a*, *b*, *c* the intersections of the circles passing through the points A, B, C respectively; then the lines A*a*, B*b*, and C*c* meet in the same point. For if the points C, P, *c* are not in a straight line, let CP meet the circle A*ca* in *n* and the circle B*cb* in *m*; then (Euc. 35. III.) $AP \times Pa = BP \times Pb = CP \times Pm$, also $AP \times Pa = CP \times Pn$; therefore $CP \times Pm = CP \times Pn$, or $Pm = Pn$, which is impossible; therefore the points *c*, *m*, *n* coalesce, and consequently the points C, P, *c* are in a straight line, and A*a*, B*b*, C*c* meet in the same point.



2. Let A, B and C be the centres of the given circles, and DE the line in which the tangents intersect, then if W be the radius of the circle whose centre is A, N the rad. of that whose centre is B, and M the radius of the other; we have, by similar triangles, $M - N : W + N :: N : BE$, and $N - M : N + M :: N : BD$. But, since $M \times W = N^2$, $W - N = \frac{N}{M}(N - M)$ and $W + M = \frac{N}{M}(N + M)$, therefore $W - N : W + N :: N - M : N + M$, and, consequently, $BE = BD = N \times \frac{N + M}{N - M}$: therefore $DE^2 = DB^2 + EB^2 - 2DB \times EB \times \cos DBE = DB^2(2 - 2\cos DBE) = DB^2(2 + 2\cos ABC)$, But

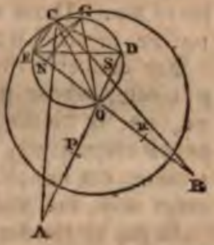
$AB = N + \frac{N^2}{M}$, $BC = N + M$, $CA = M + \frac{N^2}{M}$, $2 \cos ABC = \frac{AB^2 + BC^2 - AC^2}{AB \times BC} = \frac{N^2 + M^2}{(N + M)^2}$, $2 + 2 \cos ABC = \frac{2(N^2 + NM + M^2)}{(N + M)^2}$; therefore $DE^2 = \frac{2N^2}{(N - M)^2} \times (N^2 + NM + M^2)$, or $DE^2 : N^2 + NM + M^2 :: 2N^2 : (N - M)^2$.

I am not certain however that this is the proposer's meaning, as the proposition is rather indistinctly enunciated.

VII. QUESTION 337, *by Mr.* John Cavill, *Beighton.*

From two given points, equally distant from the centre of a given circle, to draw two right lines to a point in the circumference, so that their sum or difference may be equal to a given line ?

Solution, *by* Aliquis.



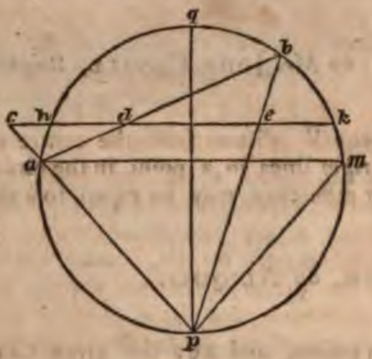
Let A and B be the given points, and ECF the given circle: Then the same construction being made as in the second solution to question 308, of the Repository, we have $AC^2 = 2AO \times PD$, and $BC^2 = 2AO \times RE$; therefore if R be put for the mean proportional between PD and RE, $AC^2 \pm 2AC \times CB + BC^2 = (AC \pm CB)^2 = 2AO(PD + RE) \pm 4AO \times R$, therefore when the sum or difference of AC, and CB is given, $PD + RE \pm 2R$ is given; but $PD + RE$ is $= 2RN$, therefore $RN \pm R$ is given: let this $= D$; then $\pm R$ is $= D - RN$, or $R^2 = D^2 - 2D \times RN + RN^2$; but R^2 is $= PD \times RE = RN^2 - NE^2 = RN^2 - EG^2 + GN^2$; theref. $D^2 - 2D \times RN = GN^2 - EG^2$, $GN^2 + 2D \times RN = D^2 + EG^2$. But RN is $= RO + ON$, and the angle NOG being given, ON has to GN a given ratio, suppose m to n; theref. $RN = RO + \frac{m}{n} \times GN$ and $GN^2 + \frac{2m}{n} \times D \times GN = D^2 + EG^2 - \frac{2m}{n} D \times RO$, therefore GN may be found by Prob. 18. V. of Simpson's Geometry, and the method of construction will be manifest.

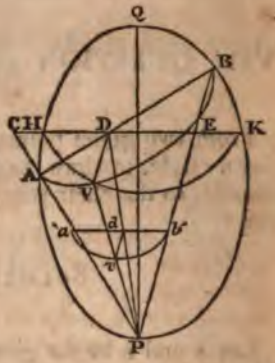
VIII. QUESTION 338, *by* L. G.

Prove that the stereographic projection of any section of a spheroid upon the plane of its equator is a circle.

SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

This elegant theorem may be readily demonstrated by the following lemma.





Let PQ be either axis of an ellipse, and HK a chord perpendicular to PQ. Draw any other chord AB cutting HK in D, and join PA, PB, meeting HK in C and E: Then shall $CD \times DE = HD \times DK$.

For conceive the ellipse to be the orthographic projection of a circle, whose diameter is $pq = PQ$; and the lines HK, AB, PC, PB, to be the orthographic projection of the lines *hk*, *ab*, *pc*, *pb* respectively: then by a well known property of this projection, CD has to *cd*, HD to *hd*, DE to *de*, and DK to *dk* the same ratio, viz. that of the conjugate to the transverse axis.

In the circle draw the chord *am* parallel to *hk* and join *pm*. The angle $pba = pma = pam = pck$, that is the angle $eba = eca$, therefore the points *e*, *b*, *c*, *a* are in the circumference of a circle, and consequently $cd \times de = ad \times db = hd \times dk$. But it has been shewn that

$$CD : cd :: HD : hd,$$

$$\text{and } DE : de :: DK : dk,$$

$$\text{therefore } CD \times DE : cd \times de :: HD \times DK : hd \times dk;$$

Hence $CD \times DE = HD \times DK$ as was to be demonstrated.

Now to apply this lemma to the question, let AVB be the line in which any plane meets the surface of the spheroid, and let $a'\,v\,b'$ be the stereographic projection of this line upon any plane perpendicular to the axis. Conceive a plane PAQB to pass along the axis, and meet the planes AVB, $a'\,v\,b'$ in AB, $a'\,b'$; Take V any point in AVB, and let a plane perpendicular to the axis pass through V, and meet the plane PAQB in the chord HK, and the surface of the spheroid in the line HVK, which will be a circle. Draw a line from D to V, which being the common section of the planes AVB, HVK, will be perpendicular to the plane PAB, and therefore perpendicular to the line HK, the diameter of the circle HVK. Draw lines from P to the points A, B, D, V meeting the projecting plane in $a'\,b'\,d$, v the projections of these points; let PA, PB meet HK in C and E, and join $d\,v$. Then, because of the parallel planes HVK, $a'\,v\,b'$, the triangles PDC, Pda' are similar: so also are the triangles PDV, Pdv, and the triangles PDE, Pdb': Hence

$$\text{CD} : a'd \;(\,::\, \text{PD} : \text{P}d)\; :: \;\text{DV} : dv,$$

$$\text{and DE} : db' \;(\,::\, \text{PD} : \text{P}d)\; :: \;\text{DV} : dv;$$

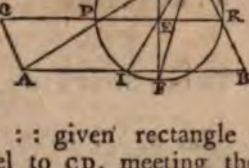
$$\text{therefore, CD} \times \text{DE} : a'd \times db' :: \text{DV}^2 : dv^2.$$

But, by the lemma, CD × DE = HD × DK, and again because HVK is a semicircle, and VD perpendicular to HK; HD × DK = DV²; therefore CD × DE = DV² and consequently $a'd \times db' = dv^2$. Now dv is perpendicular to $a'b'$, because the angles CDV, $a'dv$ are equal, and the former of these is a right angle; therefore v is in the circumference of a circle of which $a'\,b'$ is the diameter, and as this is true of the projection of every point in the line AVB; the stereographic projection of that line is a circle.

IX. QUESTION 339, *by Mr.* NOBLE, *R. M. College.*

Given the base, the difference of the angles at the base, and the rectangle of the sides, to construct the plane triangle.

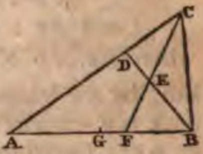
FIRST SOLUTION, *by Mr.* S. JONES, *Liverpool.*



Upon any right line HF as a diameter describe a circle, in which draw the chord FC making the angle HFC = half the given difference of the angles at the base, demit CD perpendicular to HF, and in the diameter determine the point E (Leslies Analysis B. 2. Prop. 18.) such, that HF × DE : HE × EF :: given rectangle : $(\frac{1}{2}\text{ base})^2$; through E draw REP parallel to CD, meeting the circle in R and P, in which take RQ = given base; join C, R; C, P; and from Q draw QA parallel to CR to meet CP in A, and AB parallel to QR to meet CR in B, and ACB is the required riangle.

By parallels the base $AB = QR =$ the given length, and the angle $ABC - BAC = PRC - RPC = 2\,HFC =$ the given magnitude, by the construction. Through E draw CEI to meet the base AB in I; then since PR is bisected in E, AB is also bisected in E; and by similar triangles $CP : CA :: CR : CB :: PE : AI$; therefore $CA \times CB : CP \times CR :: AI^2 : PE^2$: but $CP \times CR = HF \times DE$, and $PE^2 = HE \times EF$; wherefore by alternation, $AC \times CB : AI^2 :: HF \times DE : HE \times EF ::$ given rectangle : $(\frac{1}{2}\text{ base})^2$; but $AI = \frac{1}{2}$ base, therefore $AC \times CB =$ the given rectangle.

SECOND SOLUTION, *by Mr.* NOBLE, *the Proposer.*



Suppose it done. Bisect the vertical angle by a line CF meeting the base in F. Bisect the base in G and make the angle $DBA =$ the given difference of the angles at the base. Then by similar triangles ACF, ECB, $\frac{AC}{BC} = \frac{CF}{CE}$, but Euc. 3.6, $\frac{AC}{CB} = \frac{AF}{FB}$, therefore $\frac{CF}{CE} = \frac{AF}{FB}$. Now by Prop. B. Simson's Euc. 6 lib. $AC \times CB = AF \times FB + FC^2 = a$. Again the triangle EFB is isosceles *i. e.* $EB = FB$ because $\angle CEB = AFC$, therefore as $\angle EBF$ is given the ratio of EF to FB is given. Let $EF = m \,.\, FB$, and $EC = CF - m.\,FB$. Hence the following solution; $AF : FB :: CF : CE = CF - m.\,FB$, subtracting the consequents from the antecedents $AF : 2\,FG :: CF : m.\,FB$ therefore $CF^2 = \frac{m^2 . AF^2 . FB^2}{4\,FG^2}$. Now $AF \times FB = GB^2 - GF^2$, so finally we get, since $AF \times FB + CF^2 = a$, by putting $GF = x$, $GB = b$, $b^2 - x^2 + \frac{m^2 (b^2 - x^2)^2}{4\,x^2} = a$: this equation determines the point F, therefore make $\angle ABD =$ the given one: make $BE = BF$: join EF, then make $FB : 2\,GF :: FE : EC$. *Q. E. I.*

THIRD SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let a, b, c be the sides of the triangle, b being the given base; and A, B, C the opposite angles. By trigonometry, $\frac{a}{b} = \frac{\sin A}{\sin B}$, $\frac{c}{b} = \frac{\sin C}{\sin B}$, therefore $\frac{ac}{b^2} = \frac{\sin A \sin C}{\sin^2 B}$. But by the arithmetic of sines, $\sin A \sin C = \frac{1}{2}\cos(A - C) - \frac{1}{2}\cos(A + C)$, therefore, observing that $\sin B = \sin(A + C)$, we have

$$\frac{2ac}{b^2} = \frac{\cos(A - C) - \cos(A + C)}{\sin^2(A + C)} = \frac{\cos(A - C) - \cos(A + C)}{1 - \cos^2(A + C)}.$$

Put $A - C = \alpha$, and let β be such an angle, that $\cos\beta = \frac{2ac}{b^2}$, so that α and β are, by the question, both given; then, from the preceding equation, we have

$$\cos(A + C)\left\{1 - \cos\beta\cos(A + C)\right\} = \cos\alpha - \cos\beta,$$

or, since $\cos\alpha - \cos\beta = 2\sin\frac{1}{2}(\beta + \alpha)\sin\frac{1}{2}(\beta - \alpha)$,

$$\cos(A + C)\left\{1 - \cos\beta\cos(A + C)\right\} = 2\sin\tfrac{1}{2}(\beta + \alpha)\sin\tfrac{1}{2}(\beta - \alpha).$$

Assume now $\cos(A + C) = \frac{\cos^2\frac{1}{2}\varphi}{\cos\beta}$, then the equation may be expressed thus

$$4\cos^2\tfrac{1}{2}\varphi\sin^2\tfrac{1}{2}\varphi = 8\cos\beta\sin\tfrac{1}{2}(\beta + \alpha)\sin\tfrac{1}{2}(\beta - \alpha).$$

But $2\cos\frac{1}{2}\varphi\sin\frac{1}{2}\varphi = \sin\varphi$, therefore

$$\sin^2\varphi = 8\cos\beta\sin\tfrac{1}{2}(\beta + \alpha)\sin\tfrac{1}{2}(\beta - \alpha),$$

and $\sin\varphi = 2\sqrt{\left\{2\cos\beta\sin\frac{1}{2}(\beta + \alpha)\sin\frac{1}{2}(\beta - \alpha)\right\}}$.

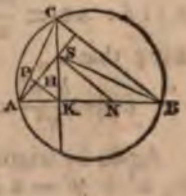
Hence φ is known, and consequently $A + C$, from the equation $\cos(A + C) = \frac{\cos^2\frac{1}{2}\varphi}{\cos\beta}$.

Now $A - C$ is given, therefore all the angles may be found, and one of the sides being also given the others may be found.

X. QUESTION 340, *by Mr.* JOHN DAWES, *Birmingham.*

Given the base and vertical angle of a plane triangle to construct it, when the segment of one side, next the angle at the base, made by a line drawn parallel to the other side from the point in the base where the perpendicular from the vertical angle cuts it, is equal to a given line.

FIRST SOLUTION, *by Mr.* DAWES, *the Proposer.*



Let ACB be the triangle; AB the base *given*, ACB the given vertical angle; and AP the given segment; join KP and draw AH at right angles to it, then will AH be given: draw SN perpendicular to AS; then AS : AK : : rad. : sin ASK, and AK : AH : : rad. : sin ASK (AKH), therefore AS : AH :: $\text{rad.}^2 : s^2$, ASK

$= \frac{\text{AH. R}^2}{\text{AS}} = s^2.\ \text{ANS} = s^2.\ \text{SNB}$, *i. e.* $(\text{R} = 1)\ \sqrt{\frac{\text{AH}}{\text{AS}}} = s.\ \text{SNB}$ $= s.(\text{NSB} + \text{NBS}) =$ (by a known theorem) $s.\ \text{NSB} \times \cos.\ \text{NBS}$ $+ \cos\ \text{NSB} \times s.\ \text{NBS} = \sqrt{\frac{\text{AH}}{\text{AS}}}$.

But $\text{AB} : s.\ \text{ASB} :: \text{AS} : s.\ \text{ABS} = s.\ \text{NBS} = \frac{s.\ \text{ASB. AS}}{\text{AB}}$; therefore $s.\ \text{NSB} \times \sqrt{(1 - \frac{s^2.\text{ASB.AS}^2}{\text{AB}^2})} + \cos\ \text{NSB} \times \frac{s.\ \text{ASB. AS}}{\text{AB}}$ $= \sqrt{\frac{\text{AH}}{\text{AS}}}$.

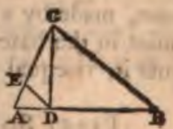
Put $\text{AH} = a$, $\text{AB} = b$, $s.\ \text{ASB} = \cos.\ \text{NSB} = d$, $s.\ \text{NSB} = g$, $\text{AS} = x$; $s.\ \text{ABS} = s.\ \text{NBS} = \frac{dx}{b}$; $\cos\ \text{NBS} = \sqrt{(1 - \frac{d^2x^2}{b^2})}$; then $g\sqrt{\left(1 - \frac{d^2x^2}{b^2}\right)} + \frac{d^2x}{b} = \sqrt{\frac{a}{x}}$: from which the unknown quantity x or AS may be found.

OBS. It is evident that it cannot be constructed geometrically, as it produces an equation of these dimensions $x^6 * - \text{A}x^4 + \text{B}x^3 + \text{C}x^2 - \text{D}x + \text{E} = 0$, unless it be when the vertical angle becomes a right angle, then sin $\text{NSB} = g$ vanishes, and $d^2 = 1$: Then the equation becomes simply

$$\frac{x}{b} = \sqrt{\frac{a}{x}},\ \text{or}\ \frac{x^2}{b^2} = \frac{a}{x},\ \text{or}\ x^3 = ab^2,$$

which is known to be true.

SECOND SOLUTION, *by* J. H. F.



Let ACB be the triangle required; $\text{AB} = b$, the given base; $\text{AE} = a$, the given segment; and $c =$ the cosine of the given angle ACB or its equal AED; $x = \text{AD}$, $y = \text{ED}$. Then by similar triangles $x : y :: b : \text{BC} = \frac{by}{x}$, and $x : a :: b : \text{AC} = \frac{ba}{x}$: but Euc. 13, 2, $\text{AB}^2 + \text{AC}^2 = 2\ \text{AB. AD} + \text{BC}^2$; that is,

$$b^2 + \frac{b^2a^2}{x^2} = 2bx + \frac{b^2y^2}{x^2}\ \text{or}\ bx^2 + ba^2 - by^2 = 2x^3.$$

Again by trigonometry $\text{AE}^2 + \text{DE}^2 - 2\ \text{AE. DE} \times c = \text{AD}^2$, that is, $a^2 + y^2 - 2ayc = x^2$, or $a^2 + y^2 - x^2 = 2ayc$: multiplying this equation by b, and adding the product to the preceding equation, we have $2\,ba^2 = 2\,x^3 + 2\,bayc$, or $y = \frac{ba^2 - x^2}{abc}$, and substituting this value for y in either of the equations, we get, after proper reduction, $x^6 - (8a^2bc^2 + 2a^4b)\,x^3 + 4a^2bc^2x^2 + 4a^4bc^2 - b^2a^4 = 0$, an equation of the sixth power, so that the problem cannot be constructed geometrically. When ACB is a right angle, or $c = 0$, the equation becomes $x^6 - 2a^2bx^3 - b^2a^4 = 0$, and the root is $x = \sqrt[3]{(ba^2)}$.

XI. QUESTION 341, *by* ANALYTICUS.

Required the fluent of $\frac{\dot{x}}{(3 - xx)\sqrt[3]{(1 - 3xx)}}$?

SOLUTION *by Mr.* CUNLIFFE.

Put $1 - 3x^2 = \left(1 - \frac{x}{z}\right)^3 = 1 - \frac{3x}{z} + \frac{3x^2}{z^2} - \frac{x^3}{z^3}$, which gives $x = \frac{3z}{2}(1 + z^2) - \frac{3z}{2}\sqrt{(z^4 + 2z^2 - \frac{1}{3})}$, $\frac{3}{2}\sqrt{(z^4 + 2z^2 - \frac{1}{3})} = \frac{3z}{2}(1 + z^2) - x$, squaring both sides,

$\frac{9z^2}{4}(z^4 + 2z^2 - \frac{1}{3}) = \frac{9z^2}{4}(1 + z^2)^2 - 3xz\,(1 + z^2) + x^2$; $x^2 = 3xz(1 + z^2) - 3z^2$; $3 - x^2 = 3 + 3z^2 - 3xz\,(1 + z^2) = 3(1 + z^2)(1 - xz)$.

Again $(1 - xz)\left(1 - \frac{x}{z}\right) = 1 - xz - \frac{x}{z} + x^2 = 1 - \frac{x}{z}(1 + z^2) + 3xz(1 + z^2) - 3z^2 = 1 - 3z^2 + \frac{x}{z}(1 + z^2)(3z^2 - 1) = (3z^2 - 1)\left\{\frac{x}{z}(1 + z^2) - 1\right\}$; $\therefore (3 - x^2)\sqrt[3]{(1 - 3x^2)} = (3 - x^2)\left(1 - \frac{x}{z}\right) = 3(1 + z^2)(3z^2 - 1)\left\{\frac{x}{z}(1 + z^2) - 1\right\}$; and hence

$$\frac{\dot{x}}{(3 - x^2)\sqrt[3]{(1 - 3x^2)}} = \frac{\dot{x}}{(3 - x^2)\left(1 - \frac{x}{z}\right)} =$$

$$\frac{\frac{1}{3}\,\dot{x}}{(1 + z^2)(3z^2 - 1)\left\{\frac{x}{z}(1 + z^2) - 1\right\}}$$

Again from the above found value of x,

$$\dot{x} = \frac{3\dot{z}}{2} \times \frac{(1+3z^2)\sqrt{(z^4+2z^2-\frac{1}{3})}-(3z^4+4z^2-\frac{1}{3})}{\sqrt{(z^4+2z^2-\frac{1}{3})}},$$

$$\tfrac{1}{3}\dot{x} = \frac{\dot{z}}{2} \times \frac{(1+3z^2)\sqrt{(z^4+2z^2-\frac{1}{3})}-(3z^4+4z^2-\frac{1}{3})}{\sqrt{(z^4+2z^2-\frac{1}{3})}},$$

$$\frac{x}{z}(1+z^2)-1 = \tfrac{3}{2}(1+z^2)^2 - \tfrac{3}{2}(1+z^2)\sqrt{(z^4+2z^2-\tfrac{1}{3})}-1$$

$$= \tfrac{1}{2}(1+6z^2+3z^4) - \tfrac{3}{2}(1+z^2)\sqrt{(z^4+2z^2-\tfrac{1}{3})};$$

and therefore $\dfrac{\frac{1}{3}\dot{x}}{\frac{x}{z}(1+z^2)-1} =$

$$\frac{\dot{z}\left\{(1+3z^2)\sqrt{(z^4+2z^2-\frac{1}{3})}-(3z^4+4z^2-\frac{1}{3})\right\}}{\left\{1+6z^2+3z^4-3(1+z^2)\sqrt{(z^4+2z^2-\frac{1}{3})}\right\}\sqrt{(z^4+2z^2-\frac{1}{3})}};$$

multiplying both the numerator and denominator of the preceding expression by $1+6z^2+3z^4+3(1+z^2)\sqrt{(z^4+2z^2-\frac{1}{3})}$, to take away the surd from the first factor of the denominator, then the denominator becomes $4\sqrt{(z^4+2z^2-\frac{1}{3})}$, and the numerator becomes $2\dot{z}(1-z^2)\sqrt{(z^4+2z^2-\frac{1}{3})} - 2\dot{z}(z^4+\frac{1}{3})$; and therefore

$$\frac{\frac{1}{3}\dot{x}}{\frac{x}{z}(1+z^2)-1} = \frac{\dot{z}(1-z^2)\sqrt{(z^4+2z^2-\frac{1}{3})}-(z^4+\frac{1}{3})\dot{z}}{2\sqrt{(z^4+2z-\frac{1}{3})}}$$

$$= \tfrac{1}{2}(1-z^2)\dot{z} - \frac{(3z^4+1)\dot{z}}{6\sqrt{(z^4+2z^2-\frac{1}{3})}} = \tfrac{1}{2}(1-z^2)\dot{z} - \frac{\frac{1}{2\sqrt{3}}(3z^4+1)\dot{z}}{\sqrt{(3z^4+6z^2-1)}},$$

and hence $\dfrac{\frac{1}{3}\dot{x}}{(1+z^2)(3z^2-1)\left\{\frac{x}{z}(1+z^2)-1\right\}} =$

$$\frac{\frac{1}{2}(1-z^2)\dot{z}}{(1+z^2)(3z^2-1)} - \frac{1}{2\sqrt{3}} \cdot \frac{(3z^4+1)\dot{z}}{(1+z^2)(3z^2-1)\sqrt{(3z^4+6z^2-1)}}$$

Now put $vz = \sqrt{(3z^4+6z^2-1)}$, $v = \dfrac{\sqrt{(3z^4+6z^2-1)}}{z}$; then

$\dot{v} = \dfrac{(3z^4+1)\dot{z}}{z^2\sqrt{(3z^4+6z^2-1)}}$, and $\dfrac{(3z^4+1)\dot{z}}{\sqrt{(3z^4+6z^2-1)}} = z^2\dot{v}$; also

$(1+z^2)(3z^2-1) = 3z^4+2z^2-1 = v^2z^2-4z^2 = z^2(v^2-4)$:

therefore $\dfrac{(3z^4+1)\dot{z}}{(1+z^2)(3z^2-1)\sqrt{(3z^4+6z^2-1)}} = \dfrac{\dot{v}}{v^2-4}.$

Again $\frac{\frac{1}{2}(1-z^2)\dot{z}}{(1+z^2)(3z^2-1)}=\frac{\frac{1}{4}\dot{z}}{3z^2-1}-\frac{\frac{1}{4}\dot{z}}{1+z^2}$, wherefore the expression is transformed to

$$\frac{\frac{1}{4}\dot{z}}{3z^2-1}-\frac{\frac{1}{4}\dot{z}}{1+z^2}-\frac{1}{2\sqrt{3}}\times\frac{\dot{v}}{v^2-4},$$

the fluents of which are

$$\frac{1}{8\sqrt{3}}\left\{\text{h. l. }\frac{z\sqrt{3}-1}{z\sqrt{3}+1}+\text{h. l. }\frac{v+2}{v-2}\right\}-\tfrac{1}{4}\text{A},$$

$=\frac{1}{8\sqrt{3}}\times\text{h. l. }\left\{\frac{z\sqrt{3}-1}{z\sqrt{3}+1}\times\frac{v+2}{v-2}\right\}-\tfrac{1}{4}\text{A}$, where A denotes the length of a circular arc to radius 1 and tangent z.

XII. QUESTION 342, *by Mr.* CUNLIFFE.

Find the sum of n terms of the progression

$$1^2.\ 2^2+3^2.\ 4^2+5^2.\ 6^2+7^2.\ 8^2\ \&c.$$

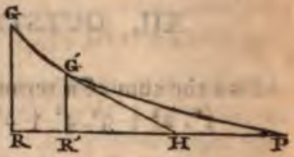
SOLUTION, *by Mr.* S. JONES, *Liverpool.*

It is manifest that the *nth* term of the series is $(2n-1)^2+4n^2$: put therefore $1^2.\ 2^2+3^2.\ 4^2+5^2.\ 6^2+7^2.\ 8^2$ &c. .. $+(2n-1)^2.\ 4n^2=\text{A}n^5+\text{B}n^4+\text{C}n^3+\text{D}n^2+\text{E}n$; and writing $n+1$ for n it becomes $1^2.\ 2^2+3^2.\ 4^2+5^2.\ 6^2+7^2.\ 8^2$ &c. . . $(2n-1)^2.\ 4n^2+(2n+1)^2.\ 4(n+1)^2=\text{A}(n+1)^5+\text{B}(n+1)^4+\text{C}(n+1)^3+\text{D}(n+1)^2+\text{E}(n+1)$. Subtracting the former expression from the latter gives $(2n+1)^2.\ 4(n+1)^2=\text{A}((n+1)^5-n^5)+\text{B}((n+1)^4-n^4)+\text{C}((n+1)^3-n^3)+\text{D}((n+1)^2-n^2)+\text{E}$, which, by expanding the terms and arranging them according to the powers of n, gives $16n^4+48n^3+52n^2+24n+4=5\text{A}n^4+(10\text{A}+4\text{B})n^3+(10\text{A}+6\text{B}+3\text{C})n^2+(5\text{A}+4\text{B}+3\text{C}+2\text{D})n+\text{A}+\text{B}+\text{C}+\text{D}+\text{E}$. Equating the homologous terms we have, $\text{A}=\frac{16}{5}$, $\text{B}=4$, $\text{C}=-\frac{4}{3}$, $\text{D}=2$, $\text{E}=\frac{2}{15}$: Hence $1^2.\ 2^2+3^2.\ 4^2+5^2.\ 6^2+7^2.\ 8^2$ &c. . . . . $(2n-1)^2\times4n^2=\frac{16n^5}{5}+4n^4-\frac{4n^3}{3}-2n^2+\frac{2n}{15}=\frac{48n^5+6n^4-20n^3-30n^2+2n}{15}=\frac{2n}{15}\times(24n^4+30n^3-10n^2-15n+1)$.

XIII. QUESTION 343, *by Mr.* CUNLIFFE.

A greyhound spied a hare at the distance of half a mile, which he immediately pursued; now the hare fled away in a right line and the greyhound made directly towards her: how far did the hare run before she was overtaken by the greyhound, supposing the greyhound's speed to that of the hare as 5 to 4, and the shortest distance from the greyhound to the line of the hare's course, a quarter of a mile?

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*



Let the points G′ and H represent contemporary places of the greyhound and hare during the chase; and let the curve GG′P represent the track of the greyhound, and the right line RHP the track of the hare; the point P denoting the place where the greyhound overtakes the hare; draw GR, G′R′ perpendicular to RHP; also draw the right line G′H.

Put $RG = a$, $RR' = x$, $R'G' = y$, tangent of $\frac{1}{2}$ $\angle R'HG' = t$, radius 1; also put curve $GG' = z$, and let the velocity of the point G′ be to that of the point H, as 1 to $n = \frac{4}{5}$.

By a known theorem $\frac{2t}{1-t^2} = \tan.\ \angle R'HG'$, and therefore $\tan \angle R'G'H = \frac{1-t^2}{2t}$, and its secant $= \frac{1+t^2}{2t}$. By the nature of the problem G′H is a tangent to the curve at G′, and hence we shall pretty obviously have $-\dot{y} : \dot{z} :: 1 : \frac{1+t^2}{2t}$, whence $\dot{z} = -\dot{y} \times \frac{1+t}{2t}$ which is proportional to the velocity of the point G in the direction of the curve GG′. Again it is pretty obvious that $R'G' : R'H :: 1 : \frac{1-t^2}{2t}$, whence $RH' = R'G' \times \frac{1-t^2}{2t}, = y \times \frac{1-t^2}{2t}$, and hence $RH = RR' + R'H = x + y \times \frac{1-t^2}{2t}$ and the fluxion of RH is $\dot{x} + \dot{y} \times \frac{1-t^2}{2t} - y\dot{t} \times \frac{1+t^2}{2t^2}$, which is also proportional to the velocity of the point H in the direction of the right line RH.

Again $-\dot{y} : \dot{x} :: 1 : \frac{1-t^2}{2t}$, whence $\dot{x} = -\dot{y} \times \frac{1-t^2}{2t}$, by means of which $\dot{x} + \dot{y} \times \frac{1-t^2}{2t} - y\dot{t} \times \frac{1-t^2}{2t^2} = -\dot{y} \times \frac{1-t^2}{2t} + \dot{y} \times \frac{1-t^2}{2t} - y\dot{t} \times \frac{1+t^2}{2t^2} = -y\dot{t} \times \frac{1+t^2}{2t^2}$; which, as just observed, is proportional to the velocity of the point H in the direction of the right line RH.

Then by the question and the nature of fluxions $\dot{z} = -\dot{y} \times \frac{1+t^2}{2t} : -y\dot{t} \times \frac{1+t^2}{2t^2} :: 1 : n$, whence $\frac{n\dot{y}}{y} = \frac{\dot{t}}{t}$, and taking the fluents h. l. (y^n) = h. l (t) : but when $y = a$, $t = m$, ∴ the correct equation of the fluents is h. l. $\frac{y^n}{a^n}$ = h. l. $\frac{t}{m}$, and ∴ $\frac{t}{m} = \frac{y^n}{a^n}$, or $t = \frac{my^n}{a^n}$, where m denotes the tangent of half the complement of the angle which the tangent at G makes with the ordinate GR. By means of what has just been deduced $\frac{1-t^2}{2t} = \frac{a^n}{2my} - \frac{my^n}{2a^n}$, hence $\dot{x} = -\dot{y} \times \frac{1-t^2}{2t} = -\frac{a^n\dot{y}}{2my^n} + \frac{my^n\dot{y}}{2a^n}$, taking the fluents $x = -\frac{a^n y^{1-n}}{2m(1-n)} + \frac{my^{1+n}}{2a^n(1+n)}$; and supposing the abscissa to commence when $y = a$, the correct equation of the curve will be $x = \frac{a}{2m(1-n)} - \frac{ma}{2(1+n)} + \frac{my^{1+n}}{2a^n(1+n)} - \frac{a^n y^{1-n}}{2m(1-n)}$, and when $y = 0$ the expression becomes $\frac{a}{2m(1-n)} - \frac{ma}{2(1+n)}$ = RP.

Again, by means of the equation $t = \frac{my^n}{a^n}$, $\frac{1+t^2}{2t} = \frac{my^n}{2a^n} + \frac{a^n}{2my}$; hence $\dot{z} = -\dot{y} \times \frac{1+t^2}{2t} = -\frac{my^n\dot{y}}{2a^n} - \frac{a^n\dot{y}}{2my^n}$, and taking the fluents $z = -\frac{my^{n+1}}{2a^n(1+n)} - \frac{a^n y^{1-n}}{2m(1-n)}$: but when $y = a$, $z = 0$, therefore the correct equation of the fluents is $z = \frac{ma}{2(1+n)} + \frac{a}{2m(1-n)} - \frac{my^{n+1}}{2a^n(1+n)} - \frac{a^n y^{1-n}}{2m(1-n)}$; and when $y=0$ the expression becomes $\frac{ma}{2(1+n)} + \frac{a}{2m(1-n)} =$ the length of the curve GG′P, the whole distance run by the greyhound before he overtook the hare.

Adapting the last expression to the example in the question, we shall have $a = \frac{1}{4}$, $n = \frac{4}{5}$ and $m = 2 - \sqrt{3} =$ the tangent of 15°, radius 1: then $a \times \left(\frac{m}{2(1+n)} + \frac{1}{2m(1-n)}\right) = a \times \frac{50+20\sqrt{3}}{9} = a \times 9{\cdot}4045573 = 2{\cdot}3511393$ miles $=$ 2 miles and 618 yards, the distance run by the greyhound, and the distance run by the hare is $\frac{4}{5} \times 2{\cdot}3511393 = 1{\cdot}881115$ miles $=$ 1 mile and 1550·7624 yards.

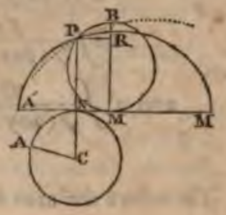
The distance run by the hare expressed in general terms is $an \times \left(\frac{m}{2(1+n)} + \frac{1}{2m(1-n)}\right)$, because the ratio of the distance run by the greyhound, is to that run by the hare as 1 to n.

The curve GG′P, has the following remarkable property, namely; The nth power of any ordinate as G′R′, is directly proportional to the tangent of half the complement of the angle R′G′H; this is manifest from the expression $\frac{y^n}{a^n} = \frac{t}{m}$, deduced in the foregoing solution.

XIV. QUESTION 344, *by* Mr. Rob. J. Dishneagh,
Trinity College, Cambridge.

If the base of a Cycloid revolve on the circumference of a circle, whose radius is equal to the radius of the generating circle. Determine the nature of the curve traced out by the extremity of the ordinate of the cycloid belonging to the point of contact, and find the area (exterior to the circle when every part of the base has been in contact with the circle.

FIRST SOLUTION, *by Mr.* ROB. J. DISHNEAGH, *the Proposer.*



Let the cycloid touch the circle in N, and every point of NA' having been in contact with NA, AN = A'N. But therefore A'NM is a tangent to the circle at N, and CN and PN therefore both perpendicular to it, they are in the same straight line; therefore P traces out a spiral about the circle; and after a whole revolution M evidently coincides with A, therefore A'M = circumference of the circle.

Now by the nature of the cycloid A'N = A'M — NM = arc PM — PR = arc versed sine $y - \sqrt{(2ay - y^2)}$.

Now let CP = radius of spiral $= u = \therefore y + a$, $\therefore y = u - a$, and AN = angle described $= t$;

therefore $t =$ arc versed sine $(u - a) - \sqrt{(2a(u-a) - (u-a)^2)}$;

therefore $\dot{t} = \dfrac{a\dot{u}}{\sqrt{(2a(u-a) - (u-a)^2)}} - \dfrac{a\dot{u} - (u - a)\dot{u}}{\sqrt{(2a(u-a) - (u-a)^2)}}$

$= \dfrac{(u-a)\dot{u}}{\sqrt{(2a(u-a) - (u-a)^2)}}$; now (Area)$\dot{}$ of the spiral $= \dfrac{u^2\dot{t}}{2a}$,

therefore the area of the spiral for half a revolution $= \displaystyle\int \frac{u^2\dot{t}}{2a} =$

$\displaystyle\int \frac{(u-a)u^2\dot{u}}{2a\sqrt{(2a(u-a) - (u-a)^2)}}$, between the values of $u = a$ and $u = 3a$; or if $y = u - a$, area for $\frac{1}{2}$ a revolution $=$

$\displaystyle\int \frac{y(y+a)^2\dot{y}}{2a\sqrt{(2ay - y^2)}}$ between the values of $y = 0$ and $y = 2a =$

$\displaystyle\int \frac{(y^3 + 2ay^2 + a^2y)\dot{y}}{2a\sqrt{(2ay - y^2)}}$ between the same values.

Now $\displaystyle\int \frac{y^3\dot{y}}{\sqrt{(2ay - y^2)}} = \frac{-y^2\sqrt{(2ay - y^2)}}{3} + \frac{5a}{3}\int \frac{y^2\dot{y}}{\sqrt{(2ay - y^2)}}$

and the first vanishes when $y = 0$ and when $y = 2a$; therefore

$\displaystyle\int \frac{y^3\dot{y}}{\sqrt{(2ay - y^2)}} = \frac{5a}{3}\int \frac{y^2\dot{y}}{\sqrt{(2ay - y^2)}}$:

In the same way $\displaystyle\int \frac{y^2\dot{y}}{\sqrt{(2ay - y^2)}} = \frac{3a}{2}\int \frac{y\dot{y}}{\sqrt{(2ay - y^2)}}$, and

$\displaystyle\int \frac{y\dot{y}}{\sqrt{(2ay - y^2)}} = a\int \frac{\dot{y}}{\sqrt{(2ay - y^2)}} = \int \frac{a\dot{y}}{\sqrt{(2ay - y^2)}} =$ circular arc radius a and versed sine $= y =$ arc PM $=$ for $\frac{1}{2}$ a revolution $=$ semi-circumference BPM.

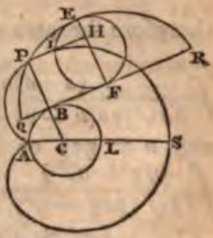
Therefore $\int \frac{y^3\dot{y}}{\sqrt{(2ay-y^2)}} = \frac{5a}{3} \cdot \frac{3a}{2} \cdot \text{BPM} = \frac{5a^2}{2} \cdot \text{BPM}.$

$$2a\int \frac{y\dot{y}}{\sqrt{(2ay-y}} = 2a \cdot \frac{3a}{2} \cdot \text{BPM} = 3a^2 \cdot \text{BPM}.$$

and $a^2\int \frac{y\dot{y}}{\sqrt{(2ay-y^2)}} = a^2 \cdot \text{BPM}.$

Therefore the area for $\frac{1}{2}$ a revolution $= \frac{\left(\frac{5a^2}{3} + 3a^2 + a^2\right)\text{BPM}}{2a} =$ $\frac{6\frac{1}{2}a^2}{2a} \times \text{BPM} = 3\frac{1}{4}a \times \text{BPM} = 3\frac{1}{4} \times$ area of the generating circle. Therefore the area for a complete revolution $= 6\frac{1}{2} \times$ area of the generating circle. Therefore the exterior part $= 5\frac{1}{2} \times$ area of the generating circle.

SECOND SOLUTION, *by Mr.* S. JONES, *Liverpool,*



Let QPER be the cycloid, F the middle of its base; EF, the diameter of its generating circle $=$ 2AC, the radius of the circle about which it revolves; CBP an ordinate through the point of contact B, and APSA the locus of P; draw PH parallel to QR cutting the circle in I, and put EF $= 2$AC $= 2a$, HF $= x$, CP $= y$, and the arc AB $=$ QB $= z$; then $y = a + x$, and, by the property of the cycloid, QB $=$ arc FI $-$ HI, therefore $\dot{z} =$ $\frac{a\dot{x}}{\sqrt{(2ax-x^2)}} - \frac{a\dot{x} - x\dot{x}}{\sqrt{(2ax-x^2)}} = \frac{x\dot{x}}{\sqrt{(2ax-x^2)}}$, and the fluxion for the area CAPC $= \frac{y^2\dot{z}}{2a} = \frac{(a+x)^2}{2a} \times \frac{x\dot{x}}{\sqrt{(2ax-x^2)}} =$ $\frac{\frac{1}{2}ax\dot{x}}{\sqrt{(2ax-x^2)}} \times \frac{2x^2\dot{x}}{\sqrt{(2ax-x^2)}} - \frac{x\dot{x}}{2a}\sqrt{(2ax-x^2)}$, and by taking the fluents we have $\int \frac{\frac{1}{2}ax\dot{x}}{\sqrt{(2ax-x^2)}} =$ circular sector CABC $- \frac{1}{2}a\sqrt{(2ax-x^2)}$, $\int \frac{2x^2\dot{x}}{\sqrt{(2ax-x^2)}} = 6 \times$ circular sector CABC $-(3a+x)\sqrt{(2ax-x^2)}$, and

$\int \frac{x\dot{x}}{2a} \times \sqrt{(2ax - x^2)} = \frac{1}{2}$ segment FIHF $- \frac{2ax - x^2}{6a} \times \sqrt{(2ax - x^2)}$; whence the area CAPC = sum of the first and second, minus the third $= 7 \times$ sector CABC $- \frac{1}{2}$ circular segment FIHF $- \frac{21a^2 + 4ax + x^2}{6a} \times \sqrt{(2ax - x^2)}$; wherefore the area APBA $= 6 \times$ sector CABC $- \frac{1}{2}$ segment FIHF $- \frac{21a^2 + 4ax + x^2}{6a} \times \sqrt{(2ax - x^2)}$; but when P comes to S, then I coincides with E, and the segment vanishes; therefore, the area of the semicurve APSLBA $= 5\frac{1}{2}$ times the semicircle ALBA, consequently the area of the whole curve, exterior of the circle, is $5\frac{1}{2}$ times its generating circle.

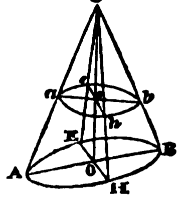
Because the cycloid is given = 3 times its generating circle, we may obtain the same result rather more concisely: For, AB = QB, and $\text{CP}^2 = (\text{CB} + \text{BP})^2 = \text{CB}^2 + 4\text{CB} \times \text{BP} - (2\text{CB} \times \text{BP} - \text{BP}^2) = \text{CB}^2 + 4\text{CB} \times \text{BP} - \text{HI}^2$, the fluxion for the area $= \frac{\text{CP}^2 \times (\text{AB})^{\cdot}}{2\text{CB}} = \frac{1}{2}\text{CB} \times (\text{AB})^{\cdot} + 2\text{BP} \times (\text{AB})^{\cdot} - \frac{\text{HI}^2}{2\text{CB}} \times \left\{ (\text{IF})^{\cdot} - (\text{HI})^{\cdot} \right\} = \frac{1}{2}\text{CB} \times (\text{AB})^{\cdot} + 2\text{BP} \times (\text{QB})^{\cdot} - \frac{1}{2}\text{HI} \times (\text{HF})^{\cdot} + \frac{\text{HI}^2}{2\text{CB}} \times (\text{HI})^{\cdot}$; the fluents of which give the area APCA = circular sector ABC + 2 cycloidal segment QBP $- \frac{1}{2}$ segment FHI $+ \text{HI}^3 \div 6\text{CB}$; and the whole area, exterior of the circle, $= 2 \times$ cycloid minus its generating semicircle $= 5\frac{1}{2}$ times the generating circle, which was required.

XV. QUESTION 345, *by* AMICUS.

Divide a given oblique cone into two equal parts, so that the area of the section made by the cutting plane may be a minimum?

SOLUTION, *by* AMICUS, *the Proposer.*

Let ACB be the given cone, and *aebh* the required section. Conceive a plane to pass through the longest and shortest sides of the cone, and meet the base in the diameter AB, and the section in the line *ab*. Then it is obvious that the terminations of the greater axis of the elliptic section will be in the lines AC and CB, or that *ab* will be the transverse axis of the ellipse. Let CO be drawn to bisect the angle ACB, meeting the base of the cone in O,



and the cutting plane in o, then because the content of the part aCb, cut off, is a constant quantity, namely half the given cone, and the area of the base $aebh$ a minimum, the perpendicular from C upon the plane $aebh$ must be a maximum, and this will evidently be the case, when the cutting plane is perpendicular to Co; for it is obvious that the perpendicular drawn from C to any other section of equal area with $aebh$ must be less than the perpendicular CO. Let a plane pass through CO and cut the base AEBH and the plane $aebh$ in EH and eh, perpendicular to AB and ab, and put $x = a\mathrm{C}$, $c = a\mathrm{CO}$ half the angle of the cone, $d =$ AB the diameter of the base, $m =$ EO $=$ OH, $n =$ OC, $h =$ the perpendicular from C upon the base AEBH, and $p = {\cdot}7854$; then $ao = x \sin c$, and $\mathrm{C}o = x \cos c$, and by similar triangles, CO : OE :: $\mathrm{C}o : oe = \frac{m}{n} x \cos c$; therefore the area of the ellipse $aebh$ is $= \frac{4m}{n} px^2 \sin c \cos c$, and the content of the cone $a\mathrm{C}b = \frac{4m}{3n} px^3 \sin c \cos^2 c$; but the content of the whole cone ACB is $= \frac{1}{3} hpd^2$, therefore $\frac{4m}{3n} px^3 \sin c \cos^2 c = \frac{1}{6} hpd^2$, and consequently $x^3 = \frac{nhd^2}{8m \sin c \cos^2 c}$, or if a and b be put for the slant sides AC and BC respectively, we have (because AO : OB :: AC : CB and AC × CB $=$ AO × OB $+$ OC^2)

$$\frac{n}{m} = \sqrt{\frac{(a+b)^2 - d^2}{d^2}}, \text{ therefore}$$

$$x^3 = \sqrt{((a+b)^2 - d^2)} \times \frac{hd}{8 \sin c \cos^2 c}.$$

Therefore if $\mathrm{C}a$ and $\mathrm{C}b$ be taken each equal to the cube root of this quantity, the points a and b, which are the extremities of the transverse axis, will be determined.

XVI. QUESTION 346, *by Mr.* P. BARLOW, *Royal Military Academy.*

Admitting the masses, distances, and densities, of the planets to be accurately known, it is required to shew a true method of ascertaining whether any law has place between the powers or roots of those quantities, similar to that which obtains with respect to their distances and times of periodic revolution?

SOLUTION, *by Mr.* P. BARLOW, *the Proposer.*

Let M, M, m, μ represent the masses of any four planets,
R, R, r, ρ their mean distances, and
D, D, d, δ their computed densities.

Let also the required law be

$$\mathrm{R}^n\mathrm{D}^m : \mathrm{M}^p :: R^nD^m : M^p$$
$$\mathrm{R}^n\mathrm{D}^m : \mathrm{M}^p :: r^nd^m : m^p$$
$$\mathrm{R}^n\mathrm{D}m : \mathrm{M}^p :: \rho^n\delta^m : \mu^p$$

Then

$$\mathrm{R}^n\mathrm{D}^mM^p = \mathrm{M}^pR^nD^m$$
$$\mathrm{R}^n\mathrm{D}^mm^p = \mathrm{M}^pr^nd^m$$
$$\mathrm{R}^n\mathrm{D}^m\mu^p = \mathrm{M}^p\rho^n\delta^m$$

making now, for the sake of abbreviation, each of these quantities to represent their respective logarithms, we have

$$n\mathrm{R} + m\mathrm{D} + pM = p\mathrm{M} + nR + mD$$
$$n\mathrm{R} + m\mathrm{D} + pm = p\mathrm{M} + nr + md$$
$$n\mathrm{R} + m\mathrm{D} + p\mu = p\mathrm{M} + n\rho + m\delta$$

or,

$$n\frac{\mathrm{R}}{R} + m\frac{\mathrm{D}}{D} = p\frac{\mathrm{M}}{M}.$$
$$n\frac{\mathrm{R}}{r} + m\frac{\mathrm{D}}{d} = p\frac{\mathrm{M}}{m}.$$
$$n\frac{\mathrm{R}}{\rho} + m\frac{\mathrm{D}}{\delta} = p\frac{\mathrm{M}}{\mu}.$$

Whence, again,

$$n\frac{\mathrm{R}}{R}\cdot\frac{\mathrm{M}}{m} + m\frac{\mathrm{D}}{D}\cdot\frac{\mathrm{M}}{m} = n\frac{\mathrm{R}}{r}\cdot\frac{\mathrm{M}}{M} + m\frac{\mathrm{D}}{d}\cdot\frac{\mathrm{M}}{M},$$
$$n\frac{\mathrm{R}}{r}\cdot\frac{\mathrm{M}}{\mu} + m\frac{\mathrm{D}}{d}\cdot\frac{\mathrm{M}}{\mu} = n\frac{\mathrm{R}}{\varrho}\cdot\frac{\mathrm{M}}{m} + m\frac{\mathrm{D}}{\delta}\cdot\frac{\mathrm{M}}{m}.$$

Or, transposing

$$n\left\{\frac{\mathrm{R}}{R}\cdot\frac{\mathrm{M}}{m} - \frac{\mathrm{R}}{r}\cdot\frac{\mathrm{M}}{M}\right\} = m\left\{\frac{\mathrm{D}}{d}\cdot\frac{\mathrm{M}}{M} - \frac{\mathrm{D}}{D}\cdot\frac{\mathrm{M}}{m}\right\}$$
$$n\left\{\frac{\mathrm{R}}{r}\cdot\frac{\mathrm{M}}{\mu} - \frac{\mathrm{R}}{\rho}\cdot\frac{\mathrm{M}}{m}\right\} = m\left\{\frac{\mathrm{D}}{\delta}\cdot\frac{\mathrm{M}}{m} - \frac{\mathrm{D}}{d}\cdot\frac{\mathrm{M}}{\mu}\right\}.$$

Whence, we have this equality,

$$\left\{\frac{\mathrm{R}}{R}\cdot\frac{\mathrm{M}}{m}-\frac{\mathrm{R}}{r}\cdot\frac{\mathrm{M}}{M}\right\}\times\left\{\frac{\mathrm{D}}{\delta}\cdot\frac{\mathrm{M}}{m}-\frac{\mathrm{D}}{d}\cdot\frac{\mathrm{M}}{\mu}\right\}=$$

$$\left\{\frac{\mathrm{R}}{r}\cdot\frac{\mathrm{M}}{\mu}-\frac{\mathrm{R}}{\rho}\cdot\frac{\mathrm{M}}{m}\right\}\times\left\{\frac{\mathrm{D}}{d}\cdot\frac{\mathrm{M}}{M}-\frac{\mathrm{D}}{D}\cdot\frac{\mathrm{M}}{m}\right\}.$$

Which must necessarily obtain if any such law as that we have supposed have place between those quantities; and this equality, it is also obvious, is totally independent of the law itself; and since we are at liberty to assume the mass, distance, and density of any particular planet equal to unity, the logarithm of which is 0, we may simplify the above by making M, R and D each equal to unity, or their logarithms = 0; for then the above becomes simply

$$\left\{\mathrm{R}m-rM\right\}\times\left\{\delta m-d\mu\right\}=$$

$$\left\{r\mu-\rho m\right\}\times\left\{dM-Dm\right\}.$$

Observing only, that these symbols are the representatives of the logarithms of the quantities they were first taken to denote; that is

M, m, μ the logarithm of the masses,
R, r, ρ the logarithm of the distances,
D, d, δ the logarithm of the densities.

In order now to submit this equality to a proper trial, let us take four planets of which those quantities are supposed to be the best known, which are Venus, the Earth, Jupiter, and Saturn, assuming also the mass, distance, and density of the earth as unity. Then adopting La Place's results, we shall have

$$\text{Masses}\left\{\begin{array}{lll}\text{Venus} = \dfrac{329630}{383137}; & \text{log.} = -\,{\cdot}06537 & = M.\\ \text{Jupiter} = \dfrac{329630}{1067{\cdot}08}; & \text{log.} = +\,2{\cdot}48983 & = m.\\ \text{Saturn} = \dfrac{329630}{3359{\cdot}40}; & \text{log.} = +\,1{\cdot}99177 & = \mu.\end{array}\right.$$

$$\text{Distances}\left\{\begin{array}{lll}\text{Venus} = {\cdot}723332; & \text{log.} = -\,{\cdot}14066 & = R.\\ \text{Jupiter} = 5{\cdot}202792; & \text{log.} = +\,{\cdot}71624 & = r.\\ \text{Saturn} = 9{\cdot}540724; & \text{log.} = +\,{\cdot}97958 & = \rho.\end{array}\right.$$

$$\text{Densities}\left\{\begin{array}{lll}\text{Venus} = 1{\cdot}024; & \text{log.} = +\,{\cdot}0103 & = D.\\ \text{Jupiter} = 0{\cdot}20093; & \text{log.} = -\,{\cdot}69696 & = d.\\ \text{Saturn} = 0{\cdot}10349; & \text{log.} = -\,{\cdot}98510 & = \delta.\end{array}\right.$$

Now applying these numbers to the preceding formula, it will be found, that the required equality does not obtain; whence we may conclude with certainty, that no such law as that we have supposed, has place with regard to the powers, or roots, of the masses, distances, and densities of the planetary bodies, at least if their present computed masses and densities be accurately determined.

XVII. QUESTION 347, *by Mr.* S. H. CHRISTIE, *Royal Military Academy.*

Let $P = \cos\frac{1}{n}a \times \sin^3\frac{1}{n}a,$

$P' = \cos\frac{1}{n}a \times \cos\frac{1}{n^2}a \times n\sin^3\frac{1}{n^2}a,$

$P'' = \cos\frac{1}{n}a \times \cos\frac{1}{n^2}a \times \cos\frac{1}{n^3}a \times n^2\sin^3\frac{1}{n^3}a,$

$P''' =$ &c.

$Q = \cos\frac{1}{n}a \times \sin^5\frac{1}{n}a,$

$Q' = \cos\frac{1}{n}a \times \cos\frac{1}{n^2}a \times n\sin^5\frac{1}{n^2}a,$

$Q'' = \cos\frac{1}{n}a \times \cos\frac{1}{n^2}a \times \cos\frac{1}{n^3}a \times n^2\sin^5\frac{1}{n^3}a,$

$Q''' =$ &c.

$R = \cos\frac{1}{n}a \times \sin^7\frac{1}{n}a,$

$R' = \cos\frac{1}{n}a \times \cos\frac{1}{n^2}a \times n\sin^7\frac{1}{n^2}a,$ &c.

Shew that

$$a = \frac{\sin a + \frac{n(n^2-4)}{2.3}(P' + P'' + \&c.) - \frac{n(n^2-4)(n^2-16)}{2.3.4.5}(Q + Q' + Q'' + \&c.) + \&c.}{\cos\frac{1}{n}a \,.\, \cos\frac{1}{n^2}a \,.\, \cos\frac{1}{n^3}a \,.\, \cos\frac{1}{n^4}a \,.\, \cos\frac{1}{n^5}a \,.\, \&c.}$$

when n is an even number; and that when n is an odd number,

$$a = \sin a + \frac{n(n^2-1)}{2.3}\left(\sin^3\frac{1}{n}a + n\sin^3\frac{1}{n^2}a + n^2\sin^3\frac{1}{n^3}a + \&c.\right)$$
$$- \frac{n(n^2-1)(n^2-9)}{2.3.4.5}\left(\sin^5\frac{1}{n}a + n\sin^5\frac{1}{n^2}a + n^2\sin^5\frac{1}{n^3}a + \&c.\right) + \&c.$$

SOLUTION, *by Mr.* CHRISTIE, *the Proposer.*

If $\underset{3}{A} = \dfrac{n.(n^2-1)}{2.3}$, $\underset{5}{A} = \dfrac{n.(n^2-1).(n^2-9)}{2.3.4.5}$, &c.

$\overset{3}{A} = \dfrac{n.(n^2-4)}{2.3}$, $\overset{5}{A} = \dfrac{n.(n^2-4).(n^2-16)}{2.3.4.5}$, &c.

$\overset{1}{P} = \cos\frac{1}{n}a$, $\overset{2}{P} = \cos\frac{1}{n}a \times \cos\frac{1}{n^2}a$,

$\overset{3}{P} = \cos\frac{1}{n}a \times \cos\frac{1}{n^2}a \times \cos\frac{1}{n^3}a$, &c.

Then

$$a = \sin a + \underset{3}{A}\left(\sin^3\frac{1}{n}a + n\sin^3\frac{1}{n^2}a + n^2\sin^3\frac{1}{n^3}a + \&c.\right)$$
$$-\underset{5}{A}\left(\sin^5\frac{1}{n}a + n\sin^5\frac{1}{n^2}a + n^2\sin^5\frac{1}{n^3}a + \&c.\right)$$
$$+ \&c.$$

when n is an odd number : and when n is an even number

$$a = \frac{\left\{\begin{array}{l}\sin a + \overset{3}{A}\left(\overset{1}{P}\sin^3\frac{1}{n}a + \overset{2}{P}n\sin^3\frac{1}{n^2}a + \&c.\right)\\ \qquad - \overset{5}{A}\left(\overset{1}{P}\sin^5\frac{1}{n}a + \overset{2}{P}n\sin^5\frac{1}{n^2}a + \&c.\right) + \&c.\end{array}\right\}}{\overset{m}{P}},$$

which are equivalent to the forms given in the question.

When n is an odd number,

$$\sin nx = n\sin x - \underset{3}{A}\sin^3 x + \underset{5}{A}\sin^5 x - \&c.$$

Let $x = \frac{1}{n}a$, then

$$\sin a = n\sin\frac{1}{n}a - \underset{3}{A}\sin^3\frac{1}{n}a + \underset{5}{A}\sin^5\frac{1}{n}a - \&c.$$

or $n\sin\frac{1}{n}a - \sin a = \underset{3}{A}\sin^3\frac{1}{n}a - \underset{5}{A}\sin^5\frac{1}{n}a + \&c.$

Substitute $\frac{1}{n}a$ for a, and we shall have

$$n^2\sin\frac{1}{n^2}a - n\sin\frac{1}{n}a = n\left(\underset{3}{A}\sin^3\frac{1}{n^2}a - \underset{5}{A}\sin^5\frac{1}{n^2}a + \&c.\right)$$

In the same manner,

$$n^3\sin\frac{1}{n^3}a - n^2\sin\frac{1}{n^2}a = n^2(\underset{3}{A}\sin^3\frac{1}{n^3}a - \underset{5}{A}\sin^5\frac{1}{n^3}a + \&c.)$$

&c. &c.

$$n^m\sin\frac{1}{n^m}a - n^{m-1}\sin\frac{1}{n^{m-1}}a = n^{m-1}(\underset{3}{A}\sin^3\frac{1}{n^m}a - \underset{5}{A}\sin^5\frac{1}{n^m}a + \&c.)$$

Adding these values, we have

$$n^m\sin\frac{1}{n^m}a - \sin a = \underset{3}{A}(\sin^3\frac{1}{n}a + n\sin^3\frac{1}{n^2}a \ldots + n^{m-1}\sin^3\frac{1}{n^m}a)$$

$$- \underset{5}{A}(\sin^5\frac{1}{n}a + n\sin^5\frac{1}{n^2}a \ldots + n^{m-1}\sin^5\frac{1}{n^m}a)$$

$$+ \&c.$$

If now m be indefinitely increased, the limit of $\sin\frac{1}{n^m}a$ is $\frac{1}{n^m}a$: and therefore $n^m\sin\frac{1}{n^m}a = a$. Consequently

$$a = \sin a + \underset{3}{A}(\sin^3\frac{1}{n}a + n\sin^3\frac{1}{n^2}a + \&c.)$$

$$- \underset{5}{A}(\sin^5\frac{1}{n}a + n\sin^5\frac{1}{n^2}a + \&c.).$$

When n is an even number,

$$\sin nx = \cos x\,(n\sin x - \overset{3}{A}\sin^3 x + \overset{5}{A}\sin^5 x - \&c.)$$

or if $x = \frac{1}{n}a$

$$\sin a = \overset{1}{P}(n\sin\frac{1}{n}a - \overset{3}{A}\sin^3\frac{1}{n}a + \overset{5}{A}\sin^5\frac{1}{n}a + \&c.)$$

theref. $\overset{1}{P}n\sin\frac{1}{n}a - \sin a = \overset{1}{P}(\overset{3}{A}\sin^3\frac{1}{n}a - \overset{5}{A}\sin^5\frac{1}{n}a + \&c.)$

Consequently,

$$\overset{2}{P}n^2\sin\frac{1}{n^2}a - \overset{1}{P}n\sin\frac{1}{n}a = \overset{1}{P}n.(\overset{3}{A}\sin^3\frac{1}{n^2}a - \overset{5}{A}\sin^5\frac{1}{n^2}a + \&c.)$$

$$\overset{3}{P}n^3\sin\frac{1}{n^3}a - \overset{2}{P}n^2\sin\frac{1}{n^2}a = \overset{3}{P}n^2(\overset{3}{A}\sin^3\frac{1}{n^3}a - \overset{5}{A}\sin^5\frac{1}{n^3}a + \&c.)$$

&c. &c.

$$\overset{m}{P}n^m\sin\frac{1}{n^m}a - \overset{m-1}{P}n^{m-1}\sin\frac{1}{n^{m-1}}a =$$

$$\overset{m}{P}n^{m-1}(\overset{3}{A}\sin^3\frac{1}{n^m}a - \overset{5}{A}\sin^5\frac{1}{n^m}a + \&c.)$$

Therefore, adding as before, $\overset{m}{P}n^m\sin\frac{1}{n^m}a - \sin a =$

$$= \overset{3}{A}(\overset{1}{P}\sin^3\frac{1}{n}a + \overset{2}{P}n\sin^3\frac{1}{n^2}a \ldots + \overset{m}{P}n^{m-1}\sin^3\frac{1}{n^m}a)$$

$$- \overset{5}{A}(\overset{1}{P}\sin^3\frac{1}{n}a + \overset{2}{P}n\sin^5\frac{1}{n^2}a \ldots + \overset{m}{P}n^{m-1}\sin^5\frac{1}{n^m}a)$$

$$+ \&c.$$

Consequently, reasoning as before, we have

$$a = \frac{\left\{\begin{array}{l}\sin a + \overset{3}{A}(\overset{1}{P}\sin^3\frac{1}{n}a + \overset{2}{P}n\sin^3\frac{1}{n^2}a + \&c.) \\ \qquad - \overset{5}{A}(\overset{1}{P}\sin^5\frac{1}{n}a + \overset{2}{P}n\sin^5\frac{1}{n^2}a + \&c.) + \&c.\end{array}\right\}}{\overset{m}{P}},$$

In particular cases where n is small, some of these values of a are very simple: thus, if $n = 2$

$$a = \frac{\sin a}{\cos\frac{1}{2}a \,.\, \cos\frac{1}{2^2}a \,.\, \cos\frac{1}{2^3}a \ldots\ldots}$$

which is Euler's expression. We may however obtain a different value of a, when $n = 2$: thus,

$$\sin a = 2\sin\frac{1}{2}a \,.\, \cos\frac{1}{2}a \,. = 2\sin\frac{1}{2}a\,(1 - 2\sin^2\frac{1}{2^2}a)$$

$$\text{theref. } 2\sin\frac{1}{2}a - \sin a = 2^2\sin^2\frac{1}{2^2}a \,.\, \sin\frac{1}{2}a$$

$$2^2\sin\frac{1}{2^2}a - 2\sin\frac{1}{2}a = 2^3\sin^2\frac{1}{2^3}a \,.\, \sin\frac{1}{2^2}a,$$

&c. &c.

$$2^m \sin\frac{1}{2^m}a - 2^{m-1}\sin\frac{1}{2^{m-1}}a = 2^{m+1}\sin^2\frac{1}{2^{m+1}}a\sin\frac{1}{2^m}a.$$

And adding as before, $2^m \sin\frac{1}{2^m}a - \sin a =$

$$2^2\sin^2\frac{1}{2^2}a\sin\frac{1}{2}a + 2^3\sin^2\frac{1}{2^3}a\sin\frac{1}{2^2}a \ldots 2^{m+1}\sin\frac{1}{2^{m+1}}a\sin\frac{1}{2^m}a,$$

and in the same manner as before,

$$a = \sin a + 2^2\sin^2\frac{1}{2^2}a\sin\frac{1}{2}a + 2^3\sin^2\frac{1}{2^3}a\sin\frac{1}{2^2}a + \&c.$$

If $n = 3$,

$$a = \sin a + 4\left(\sin^3\frac{1}{3}a + 3\sin^3\frac{1}{3^2}a + 3^2\sin^3\frac{1}{3^3}a + \&c.\right)$$

which is remarkably simple.

If $n = 4$,

$$a = \frac{\sin a + 8\left(\cos\frac{1}{4}a\sin^3\frac{1}{4}a + \cos\frac{1}{4}a\cos\frac{1}{4^2}a\,4\sin^3\frac{1}{4^2}a + \&c\right.}{\cos\frac{1}{4}a\,.\cos\frac{1}{4^2}a\,.\cos\frac{1}{4^2}a\ldots\ldots}$$

If $n = 5$,

$$a = \sin a + 20\left(\sin^3\frac{1}{5}a + 5\sin^3\frac{1}{5^2}a + \&c.\right)$$

$$- 16\left(\sin^5\frac{1}{5}a + 5\sin^5\frac{1}{5^2}a + \&c.\right)$$

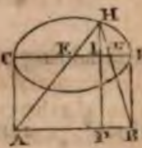
and so on for other values of n.

IX. QUESTION 348, *by Mr.* LOWRY, *R. M. College.*

Let CD be a diameter of an ellipse or hyperbola given by position, two points may be found such, that straight lines being drawn from them to any point in the curve, to meet CD in C and F; the segment EF shall be a mean proportional between CE and DF.

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

On the tangents to the curve at the extremities of the diameter CD, take CA and DB each equal to the diameter which is conjugate to CD, then are A and B the points required.



For, join AB and draw HP parallel to CA meeting CF in I, and AB in P. Then, because CA is parallel to DB and also to HI, $AC : CE :: HI : EI$, and $BD = AC : DF :: HI : IF$, therefore $AC^2 : CE \times DF :: HI^2 : EI \times IF$. Again, because AC is parallel and equal to DB, AB is parallel and equal to CD, $AP = CI$ and $BP = DI$; therefore $AB : AP :: EF : EI$, and $AB : BP :: EF : IE$; therefore $AB^2 = CD^2 : AP \times PB = CI \times ID :: EF^2 : EI \times IF$; but, by the property of the curve, $CD^2 : CI \times ID :: AC^2$ (= square of the conjugate) : HI^2; therefore $AC^2 : HI^2 :: EF^2 : EI \times IF$, but it was proved above that $AC^2 : CE \times DF :: HI^2 : EI \times IF$; therefore $AC^2 : CE \times DF :: AC^2 : EF^2$, and consequently $CE \times DF = EF^2$, as was to be demonstrated.

It may be shewn in a similar manner, that if AC^2 or DB^2 be taken to the square of the diameter which is conjugate to CD as m to n, then $CE \times DF : EF^2 :: n : m$; therefore if $AC^2 =$ twice the square of the conjugate, $2CE \times DF$ is $= EF^2$. In this case $CF^2 + DE^2$ is equal to the square of the diameter CD, which is also a curious enough theorem, and is analagous to a porism relating to the circle discovered long ago by Fermat.

XIX. QUESTION 349, *by Mr.* LOWRY.

In what curve must a body be constrained to move upon the surface of an upright cylinder, so that when urged by the force of gravity, it may descend from one given point to another in the shortest time possible?

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

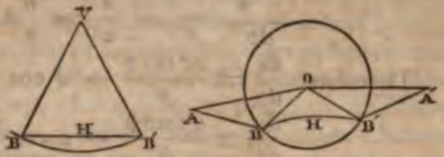
Since gravity is the only force that has any effect in accelerating the motion of the body, it is evident that the velocity at any instant, and the time requisite for describing any particular space, will be precisely the same as if the surface of the cylinder were extended on a vertical plane, and the body were to descend from one of the given points to the other on that plane. The curve of quickest descent is, in this case, a cycloid, as is well known, and all that is necessary for resolving the present question is, to trace that curve on the surface of the cylinder, which may be done by first describing the cycloid on a plane and then transfering it to the cylinder, by drawing ordinates of the proper length perpendicular to the circumference of the base.

XX. PRIZE QUESTION 350, *by* G. V.

Suppose a conical mountain of a given figure to be situated in a plane. Find the length of the shortest road between two given points, on the plane, on opposite sides of the mountain ?

SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let O be the centre of the base of the mountain, A and A′ the given points on the plane, and ABHB′A′ the shortest road from A to A′, the curve BHB′ representing that part of it which passes over the surface of the cone, and terminates in its base at B, B′, and the straight lines AB, A′B′ the two remaining parts. Let us suppose the portion of the conical surface contained by the arc BB′, and lines drawn from B, B′ to the vertex, to be developed into a plane surface, which will be a sector of a circle VBB′, V being the centre ; then, as the line BHB′ must be the shortest possible that can be drawn on the cone from B to B′, it will manifestly be BHB′ the chord of BB′ the arc of the sector.



Join OB, OB′, and let the following notation be adopted,

BO the radius of base $= c$,

BV the slant side of cone $= \frac{c}{n}$,

$\text{AO} = a$, $\text{A}'\text{O} = a'$, $\text{AB} = p$, $\text{A}'\text{B}' = p'$,

$\angle \text{AOA}' = \alpha$, $\angle \text{AOB} = \phi$, $\text{A}'\text{OB}' = \phi'$, $\angle \text{BOB}' = \theta$.

Because, by the nature of the figure, BB′ the arc of the sector BOB′, is equal to BB′ the arc of the sector BVB′, we have

$$\text{VB}\left(\frac{c}{n}\right) : \text{OB}\ (c) :: \angle \text{BOB}'\ (\theta) : \angle \text{BVB}' = n\theta.$$

Hence, the chord $\text{BHB}' = \frac{2c}{n} \sin\left(\frac{n}{2}\theta\right)$, and the analytical expression for the whole path ABHB′A′ is

$$p + \frac{2c}{n} \sin \frac{n}{2}\theta + p',$$

which by the nature of the question must be a minimum.

Let this expression be denoted by u, then u is a function of the two independent variable quantities ϕ, ϕ' ; therefore, by the theory of *Maxima* and *Minima*, we must have,

$$\frac{du}{d\phi} = 0, \quad \frac{du}{d\phi'} = 0.$$

To compute these quantities, we have

$$p^2 = a^2 + c^2 - 2ac\cos\varphi,$$
$$p'^2 = a'^2 + c^2 - 2a'c\cos\varphi',$$
$$\theta = \alpha - (\varphi + \varphi'), \text{ and therefore}$$
$$\sin\frac{n}{2}\theta = \sin\frac{n}{2}\left\{\alpha - (\varphi + \varphi')\right\}.$$

Hence, $\frac{dp}{d\varphi} = \frac{ac\sin\varphi}{p}$, $\frac{dp'}{d\varphi'} = \frac{ac\sin\varphi'}{p'}$,

$$\frac{d\sin\frac{n}{2}\theta}{d\varphi} = -\frac{n}{2}\cos\frac{n}{2}\theta = \frac{d\sin\frac{n}{2}}{d\varphi'}.$$

Therefore, $\frac{du}{d\varphi} = \frac{ac\sin\varphi}{p} - c\cos\frac{n}{2}\theta = 0,$

$$\frac{du}{d\varphi'} = \frac{a'c\sin\varphi'}{p'} - c\cos\frac{n}{2}\theta = 0.$$

But, putting A and A′ for the angles OAB and OA′B′, we have

$$\frac{a\sin\varphi}{p} = \sin(A + \varphi), \text{ and}$$
$$\frac{a'\sin\varphi'}{p'} = \sin(A' + \varphi'), \text{ therefore,}$$
$$\sin(A + \varphi) = \cos\frac{n}{2}\theta, \quad A + \varphi + \frac{n}{2}\theta = 90^\circ,$$
$$\sin(A' + \varphi') = \cos\frac{n}{2}\theta, \quad A' + \varphi' + \frac{n}{2}\theta = 90^\circ,$$
$$\sin A = \cos\left(\frac{n}{2}\theta + \varphi\right)$$
$$\sin A' = \cos\left(\frac{n}{2}\theta + \varphi'\right).$$

But $\sin A = \frac{c}{a}\sin(A + \varphi)$ and $\sin A' = \frac{c}{a'}\sin(A' + \varphi)$.

Therefore the equations of the problem are

$$\frac{c}{a}\cos\frac{n}{2}\theta = \cos\left(\frac{n}{2}\theta + \varphi\right) \ldots\ldots\ldots\ldots\ldots 1.$$
$$\frac{c}{a'}\cos\frac{n}{2}\theta = \cos\left(\frac{n}{2}\theta + \varphi'\right) \ldots\ldots\ldots\ldots 2.$$
$$\alpha - \theta = \varphi + \varphi' \ldots\ldots\ldots\ldots\ldots\ldots 3.$$

By these it would be easy to eliminate the angles φ, φ', and the result would be an equation involving the single angle θ, which being found, the angles φ, φ' would evidently be known; but it will be easier to determine θ, and thence φ and φ' by trials, by giving to θ successive values and finding that which renders $\alpha - \theta = \varphi + \varphi'$. The values of φ and φ' corresponding to any assumed value of θ may be easily determined, either by calculation, or an obvious geometrical construction.

NOTICES.

NEW BOOKS.

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON, FOR 1812.

The Mathematical Papers contained in this volume are, 1. On the grounds of the method which LaPlace has given in the second chapter of the third book of his Mécanique Celeste, for computing the attractions of spheroids of every description, by James Ivory, A. M.—2. On the Attractions of an extensive class of Spheroids, by James Ivory, A. M.—3. Observations of a Comet, with Remarks on the construction of its different parts, by William Herschell, LL.D. F.R.S.—4. Observations of a second Comet, with Remarks on its construction, by William Herschell, LL. D. F. R. S.—5. Of the Attractions of such Solids as are terminated by Planes; and of Solids of greatest Attraction, by Thomas Knight, Esq.—6. Of the Penetration of a Hemisphere, by an indefinite number of equal and similar cylinders, by Thomas Knight, Esq.—7. Observations on the Measurement of three Degrees of the Meridian conducted in England, by Lieut. Col. William Mudge, by Don Joseph Rodriguez.

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON, FOR 1813.

The Mathematical Papers contained in this volume are, 1. On a remarkable application of Cotes's Theorem, by I. F. W. Herschell, Esq.—2. Observations on the Summer Solstice, 1812, at the Royal Observatory, by John Pond, Esq. Astronomer Royal, F. R. S.—3. A Catalogue of North Polar Distances of some of the principal fixed Stars, by John Pond, Esq. Astronomer Royal, F. R. S.—4. Observations of the Winter Solstice, of 1812, with the Mural Circle at Greenwich, by John Pond, Esq. Astronomer Royal, F. R. S.—5. Catalogue of North Polar Distances of eighty-four principal fixed Stars, deduced from observations made with the Mural Circle at the Royal Observatory, by John Pond, Esq. Astronomer Royal, F. R. S.—6. Observations of the Summer Solstice, 1813, with the Mural Circle at the Royal Observatory, by John Pond, Esq. Astronomer Royal, F. R. S.

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON, FOR 1814.

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* *Sabots* are cylindrical pieces of heart of oak, attached to a cannon ball in some artillery experiments, in order to prevent the rolling of the ball in the bore of the piece.

ARTICLE II.

Solutions to Questions proposed in Number XIII.

I. QUESTION 351, *by Mr.* JOHN BAINES, *jun. Reading.*

To divide a given number into two parts, such that the sum of their squares, and the difference of their cubes may be squares?

SOLUTION, *by Mr.* J. BAINES, *the Proposer.*

First, to find two numbers such that the sum of their squares and the difference of their cubes may be squares.

Let x denote the greater, and y the less;

$$\text{assume } x = (r^2 - s^2)\,v \text{ and } y = 2rsv,$$

then $x^2 + y^2$ will be a square $= (r^2 + s^2)^2 v^2$,

$$\text{and } x^3 - y^3 = \{(r^2 - s^2)^3 - 8r^3s^3\}\,v^3;$$

$$\text{therefore } \{(r^2 - s^2)^3 - 8r^3s^3\}\,v$$

must be a square, which it evidently will be when v is taken $= (r^2 - s^2)^3 - 8r^3s^3$; therefore

$$x = (r^2 - s^2) \times \{(r^2 - s^2)^3 - 8r^3s^3\}$$
$$y = \quad 2rs \quad \times \{(r^2 - s^2)^3 - 8r^3s^3\}$$

where r and s may be any numbers taken at pleasure.

To resolve the question it is therefore only necessary to make

$$x + y \text{ or } (r^2 - s^2 + 2rs) \times \{(r^2 - s^2)^3 - 8r^3s^3\}$$

equal to the given number n, but this cannot be done generally for any value of n, but only for particular values which depend entirely on the assumed values of r and s.

When x and y are numbers found as above, it is evident that xu^2 and yu^2 (u^2 being any square number at pleasure) will also satisfy the second and third conditions; let these quantities, therefore be assumed for the parts required, then $xu^2 + yu^2 = n$, and $u^2 = \frac{n}{x+y}$, or

$$u^2 = \frac{n}{(r^2 - s^2 + 2rs) \times \left\{(r^2 - s^2)^3 - 8r^3s^3\right\}}.$$

Therefore the question can always be resolved when the given number divided by the assumed denominator produces a square number.

If r be taken $= 2$ and $s = 1$, then $x = 3 \times -37$ and, $y = 4 \times -37$, or changing the signs, $x = 111$ and $y = 148$; therefore, in this case, if n be of the form $(x + y) \times u^2$ or $259u^2$, $148u^2$ and $111u^2$ are the parts required.

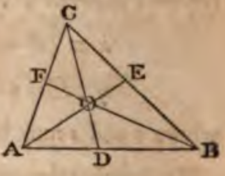
II. QUESTION 352, *by* REDINTEGRATOR.

Given the three right lines bisecting the angles and terminating in the opposite sides to determine the triangle.

SOLUTION, *by the Proposer*.

Let ACB represent the required triangle with the lines CD, AE, BF, bisecting the angles and terminating in the opposite sides, intersecting each other in O.

Put CD $= a$, AE $= b$, BF $= c$, AB $= x$, AC $= y$, and BC $= z$. By Prop. 3, Euc. VI. AC + BC : AB ::



AC : AD $= \frac{\text{AB} \times \text{AC}}{\text{AC} + \text{BC}} = \frac{xy}{y+z}$. Also

AC + BC : AB :: BC : BD $= \frac{\text{AB} \times \text{BC}}{\text{AC} + \text{BC}}$

$= \frac{xz}{y+z}$.

And by another known property,

AC × BC — AD × BD = CD^2, that is $yz - \frac{x^2yz}{(y+z)^2} = a^2$, or

$$yz \times \left\{(y+z)^2 - x^2\right\} = a^2 \times (y+z)^2.$$

In like manner,

$$xz \times \left\{(x+z)^2 - y^2\right\} = b^2 \times (x+z)^2 \text{ : and}$$

$$xy \times \left\{(x+y)^2 - z^2\right\} = c^2 \times (x+y)^2.$$

From the first of these equations,

$$z \times (x+y+z) = \frac{a^2 \times (y+z)^2}{y \times (y+z-x)},$$

and from the second, $z \times (x+y+z) = \dfrac{b^2 \times (x+z)^2}{x \times (x+z-y)}$;

and therefore $\dfrac{a^2 \times (y+z)^2}{y \times (y+z-x)} = \dfrac{b^2 \times (x+z)^2}{x \times (x+z-y)}$.

Put $y = vx$, and $z = ux$, by means of which, the last equation will become

$$\frac{a^2 \times (v+u)^2}{v \times (v+u-1)} = \frac{b^2 \times (1+u)^2}{1+u-v} \ldots\ldots\ldots\ldots(1).$$

Again, from the second of the foregoing equations,

$x \times (x+y+z) = \dfrac{b^2 \times (x+z)^2}{z \times (x+z-y)}$: and from the third,

$x \times (x+y+z) = \dfrac{c^2 \times (x+y)^2}{y \times (x+y-z)}$; and therefore,

$$\frac{b^2 \times (x+z)^2}{z \times (x+z-y)} = \frac{c^2 \times (x+y)^2}{y \times (x+y-z)}.$$

This latter equation by making the same substitutions for y and z as before, becomes

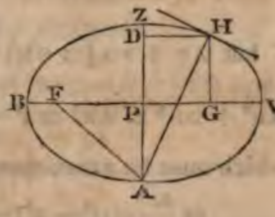
$$\frac{b^2 \times (1+u)^2}{u \times (1+u-v)} = \frac{c^2 \times (1+v)^2}{v \times (1+v-u)} \ldots\ldots(2).$$

And from equations (1) and (2) v and u must be determined, which will give the ratio of the sides of the triangle, after which the determination of the sides themselves will be very easy.

III. QUESTION 353, *by Mr.* JAMES ADAMS.

To determine that point in the curve of a given ellipse, which is at the greatest distance from the extremity of the conjugate axis?

FIRST SOLUTION, *by Mr.* JAMES ADAMS, *the Proposer.*



Let BV and AZ represent the two principal axes, and AH the required chord; draw HD perpendicular to AZ, and join A and the focus F.

It is evident that the shortest line that can be drawn from any given point to the ellipse must be perpendicular to the curve; therefore AH must be perpendicular to the curve at H. By Emerson's Conics, Prop. II. of the ellipse, we have

$$DA : DP :: PV^2 : PA^2;$$

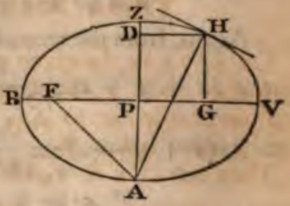
by division, $DA - DP : DA :: PV^2 - PA^2 : PV^2$;

that is, $AP : DA :: PV^2 - PA^2 : PV^2$;

therefore, $AP \times PV^2 = DA \times (PV^2 - PA^2)$, and

$$DA = \frac{AP \times PV^2}{(PV + PA)(PV - PA)} = \text{cosec } FAP \times AP \text{ (radius 1).}$$

It must be remarked that AD must be greater than AP, and less than 2AP, or AZ.



Suppose $AD = \dfrac{PV^2 \times PA}{PV^2 - PA^2} = 2AP$,

or, $AD = \dfrac{PV^2}{PV^2 - PA^2} = 2$,

from which equation $PV = PA\sqrt{2}$; or $PA : PV :: 1 : \sqrt{2}$; therefore PV must be taken greater than $PA\sqrt{2}$ to answer the conditions of the problem.

If $PV = 2AP = AZ$; then $AD = \dfrac{\frac{1}{2}PV \times PV^2}{PV^2 - \frac{1}{4}PV^2} = \frac{2}{3}PV = \frac{2}{3}AZ$.

SECOND SOLUTION, *by Mr.* JOHN BAINES, *jun.*

Let BV, AZ be the transverse and conjugate diameters of the given ellipse, and H the required point in the curve. Draw HG, HD parallel to ZP, VP, and join AH. Put $BV = 2t$, $AZ = 2c$, and $GP = HD = x$; then $VG = t - x$, and $BG = t + x$.

By the property of the ellipse,

$BV^2 : AZ^2 :: VG \times BG : GH^2$, that is

$$4t^2 : 4c^2 :: (t - x)(t + x) : \frac{c^2}{t^2}(t^2 - x^2) = GH^2;$$

$$\text{hence } GH = \frac{c}{t}\sqrt{(t^2 - x^2)}:$$

but $AP + PD (= GH) = AD = c + \dfrac{c}{t}\sqrt{(t^2 - x^2)}$, and

$$AD^2 + DH^2 = AH^2 = c^2 + \frac{2c^2}{t}\sqrt{(t^2 - x^2)} + \frac{c^2}{t^2}(t^2 - x^2) + x^2,$$

which must be a maximum, by the question,

$$\text{or } \frac{2c^2}{t}\sqrt{(t^2 - x^2)} - \frac{c^2 x^2}{t^2} + x^2 = \text{a maximum.}$$

In fluxions $\frac{-2c^2x\dot{x}}{t\sqrt{(t^2-x^2)}} - \frac{2c^2x\dot{x}}{t^2} + 2x\dot{x} = 0$, which being divided by $2x\dot{x}$ and reduced, gives $x = \frac{t^2}{t^2-c^2}\sqrt{(t^2-2c^2)}$.

Limitation. When $2c^2$ is greater than t^2, the value of x will be imaginary, which shews that the question does not admit of a maximum in that case, but the line AH will coincide with AZ.

THIRD SOLUTION, *by* RURICOLA.

Let BZVA be the given ellipse, BV the transverse and AZ the conjugate axis, P the centre, AH the line required, and draw HD perpendicular to AZ. Put $PB = PV = a$, $PA = PZ = b$, $PD = y$ and $DH = x$.

Then $AD = b + y$, $AH^2 = AD^2 + DH^2 = (b+y)^2 + x^2$;

and by the property of the ellipse $x^2 = \frac{a^2}{b^2} \times (b^2 - y^2)$,

wherefore $AH^2 = (b+y)^2 + x^2 = a^2 + b^2 + 2by - y^2 \times \frac{a^2-b^2}{b^2}$,

which is to be a maximum by the question; therefore putting the fluxion of the expression $= 0$, we shall have

$$2b\dot{y} - 2y\dot{y} \times \frac{a^2-b^2}{b^2} = 0,$$

whence $y = \frac{b^3}{a^2-b^2} = PD$: and $AD = AP + PD = \frac{a^2b}{a^2-b^2}$.

$$DH^2 = x^2 = \frac{a^4 \times (a^2 - 2b^2)}{(a^2-b^2)^2},$$

$$AH^2 = AD^2 + DH^2 = \frac{a^4}{a^2-b^2} = \frac{PB^2 \times PB^2}{PB^2 - PA^2}.$$

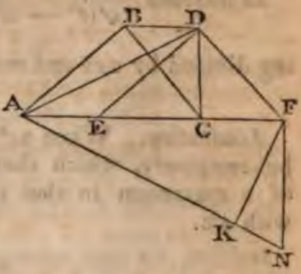
Therefore $PB^2 - PA^2 : PB^2 :: PB^2 : AH^2$.

IV. QUESTION 354, *by* ALIQUIS.

Given the two sides, and the sum of the squares of the base and perpendicular of a plane triangle, to construct it?

FIRST SOLUTION, *by* ALIQUIS, *the Proposer.*

Let ABC be the triangle required; draw CD perpendicular to AC



meeting BD, drawn parallel to the base, in D, and join AD; then CD is = the perpendicular of the triangle, and $AC^2 + CD^2 = AD^2$, therefore AD is given. Draw DE and DF parallel to BA and BC, then DE is $= AB$, and $DF = BC$, the given sides of the triangle; also AE is $= BD = CF$, consequently EC is = the difference of the segments AC, CF. Wherefore $AF \times EC = AD^2 - DF^2$ = a given space; also $AC^2 - CE^2 = AD^2 - DE^2$, or $4AC^2 - 4CE^2 = 4AD^2 - 4DE^2$ = a given space. But $2AC = AF + EC$; therefore $4AC^2 - 4EC^2 = AF^2 + 2AF \times EC - 3EC^2 = 4AD^2 - 4DE^2$ and $AF^2 - 3EC^2 = 4AD^2 - 4DE^2 - 2AD^2 + 2DF^2 = 2AD^2 - 4DE^2 + 2DF^2$ a given space; therefore $AF \times EC$ and $AF^2 - 3EC^2$ are each given spaces. Let AFK be a triangle, right angled at K, and having the side FK such that $FK^2 = 3CE^2$, then $AF^2 - 3EC^2 = AF^2 - FK^2 = AK^2$, therefore AK^2 is = the given space $2AD^2 - 4DE^2 + 2DF^2$, and consequently AK = a given line. Again, because $FK^2 = 3CE^2$, FK has to CE a given ratio, and $AF \times EC$ being given, $AF \times FK$ is given. Draw FN perpendicular to AF meeting AK produced in N; by similar triangles,

$$AK : AF :: FK : FN, \text{ or } AK \times FN = AF \times FK;$$

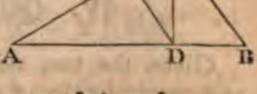
therefore FN is given, and since $FN^2 = AN \times NK$, the problem is reduced to the following, viz.

To add KN to AK, so that the rectangle under the whole, and the part added may be equal to a given space.

SECOND SOLUTION, *by* RURICOLA.

Let ACB represent the required triangle, AC and BC being the two given sides; CD perpendicular to the base AB and DE perpendicular to AC, and by the question, put

$$AB^2 + DC^2 = l^2.$$



Then

$$AB^2 + DC^2 = (AD + DB)^2 + DC^2$$
$$= AD^2 + 2AD \times DB + DB^2 + DC^2$$
$$= AD^2 + 2AD \times DB + BC^2, \text{ because } BC^2 = DB^2 + DC^2.$$

Wherefore, $AD^2 + 2AD \times DB = l^2 - BC^2$.

Now put $AC = a$, $BC = b$ and $AB = x$.

Then by obvious properties,

$$\text{AD}^2 = ax,$$
$$\text{DC}^2 = \text{AC}^2 - \text{AD}^2 = a^2 - ax,$$
$$\text{DB}^2 = \text{BC}^2 - \text{DC}^2 = b^2 - a^2 + ax = ax - (a^2 - b^2).$$
$$\text{DB} = \sqrt{\{ax - (a^2 - b^2)\}};$$

and hence,

$$\text{AD}^2 + 2\text{AD} \times \text{DB} = ax + 2\sqrt{(ax)}\sqrt{\{ax - (a^2 - b^2)\}}$$
$$= ax + 2\sqrt{\{a^2x^2 - (a^2 - b^2)ax\}} = l^2 - b^2;$$

dividing by a, $x + 2\sqrt{\left\{x^2 - \frac{(a^2 - b^2)}{a}x\right\}} = \frac{l^2 - b^2}{a}$:

and putting $\frac{a^2 - b^2}{a} = m$, and $\frac{l^2 - b^2}{a} = n$,

the preceding expression becomes

$$x + 2\sqrt{(x^2 - mx)} = n,$$

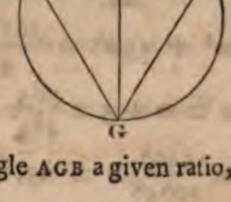
which being reduced gives

$$x = \frac{2m - n}{3} \pm \tfrac{2}{3}\sqrt{(m^2 - mn + n^2)}.$$

V. QUESTION 355, *by* Mr. S. Jones, *Liverpool.*

In a given circle to inscribe a triangle, such that the difference of its sides shall be in a given ratio to the difference of the segments of the base made by a perpendicular from the vertical angle; and the prism, whose base is the triangle and altitude the said perpendicular a maximum.

Solution, *by* Mr. Lowry, *R. M. College.*



Suppose that ACB is the triangle required, DG the given diameter drawn perpendicular to the base AB. Let EC be drawn parallel to the base to meet DG in E; then the square of half the difference of the sides is = DE × FG, and the square of half the difference of the segments of the base is DE × EG, therefore, because the ratio of the difference of the sides to the difference of the segments of the base is given, the ratio of DE × FG to DE × EG or of FG to EG is given; therefore, by division, the ratio of EF to FG is given. Wherefore the triangle ACB has to the triangle AGB a given ratio, namely, the same which the perpendicular EF has to FG; therefore when the prism EF × △ACB is a maximum, the prism FG × △AGB is a maximum, or $FG^2 \times FB$ a maximum. Let a tangent be drawn to the circle at B, to meet the diameter, produced, in N; then $FG^2 \times FB$ is a maximum when GF is equal to 2FN, as is well known; or, which comes to the same thing, when GE is equal to $\frac{5}{6}$GD. The construction of the triangle for the maximum prism is therefore obvious.

VI. QUESTION 356, *by Mr.* CUNLIFFE.

To find two rational fractions, either of which being added to the fourth power of the other shall make the same sum?

FIRST SOLUTION, *by Mr.* JOHN BAINES, *jun.*

Let $\frac{x+y}{nx}$ and $\frac{x-y}{nx}$ represent the two fractions; then

$\frac{(x+y)^4}{n^4x^4} + \frac{x-y}{nx}$ and $\frac{x+y}{nx} + \frac{(x-y)^4}{n^4x^4}$ must be equal,

or which is the same

$$(x+y)^4 + n^3x^3(x-y) = n^3x^3(x+y) + (x-y)^4;$$

by transposition,

$$(x+y)^4 - (x-y)^4 = n^3x^3 \times (x+y) - n^3x^3 \times (x-y),$$

and by reduction, $8x^3y + 8xy^3 = 2n^3x^3y$,

and dividing by $2xy$ and transposing, we have

$$n^3x^2 - 4x^2 = 4y^2, \text{ or } x^2 \times (n^3-4) = 4y^2 = \text{a square};$$

hence we have only to find such a value of n as will make $n^3 - 4 =$ a square: such a case occurs when $n = 5$, but upon inspection it appears that $n = 5$ gives $x - y =$ a negative quantity, which is inadmissible: wherefore, put $n = m + 5$, and we shall have $n^3 - 4 = m^3 + 15m^2 + 75m + 121$, to be made into a square. Assume $pm + 11$ for its root, that is,

put $m^3 + 15m^2 + 75m + 121 = (pm+11)^2 = p^2m^2 + 22pm + 121$,

and in order to make the third term disappear, take $22p = 75$,

or $p = \frac{75}{22}$; hence

$m^3 + 15m^2 = \frac{5625m^2}{484}$, or $m = -\frac{1635}{484}$, therefore

$n = m + 5 = \frac{785}{484}$, $x^2 . (n^3 - 4) = 4y^2 = \frac{30217009}{113379904}x^2 = 4y^2$,

or $\frac{5497}{10648}x = 2y$, therefore $y = \frac{5497}{21296}x$:

hence $\frac{x+y}{nx} = \frac{26793}{34540}$, and $\frac{x-y}{nx} = \frac{15799}{34540}$, are two fractions that will answer. For

$$\left(\frac{26793}{34540}\right)^4 + \frac{15799}{34540} = \frac{26793}{34540} + \left(\frac{15799}{34540}\right)^4$$

$= \frac{1166353341300328801}{1423276677734560000}$, as was required.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let the two fractions be denoted by x and y; then, by the question, $x^4 + y = y^4 + x$; whence $x^4 - y^4 = x - y$; and dividing by $x - y$, $x^3 + x^2y + xy^2 + y^3 = 1$.

Put $ny = x$ and the last equation becomes
$n^3y^3 + n^2y^3 + ny^3 + y^3 = y^3 \times (n^3 + n^2 + n + 1) = 1$;

whence $y^3 = \frac{1}{n^3 + n^2 + n + 1}$.

We have, therefore, to find such a positive value for n as will make $n^3 + n^2 + n + 1$ a rational cube.

Put $n^3 + n^2 + n + 1 = (n + r)^3 = n^3 + 3n^2r + 3nr^2 + r^3$, and take $3n^2r = n^2$, or $r = \frac{1}{3}$, in order to take away the two leading terms on each side of the equation; and then from the equality of the remaining terms, viz. $n + 1 = 3nr^2 + r^3$, we shall have $n = \frac{r^3 - 1}{1 - 3r^2} = -\frac{13}{9}$. But we want a positive value for n, therefore put $n = v - \frac{13}{9}$;

from whence $n^3 + n^2 + n + 1 = v^3 - \frac{10v^2}{3} + \frac{118v}{27} - \frac{1000}{729}$, which is to be a cube.

Put $v^3 - \frac{10v^2}{3} + \frac{118v}{27} - \frac{1000}{729} = \left(sv - \frac{10}{9}\right)^3 =$
$s^3v^3 - \frac{10s^2v^2}{3} + \frac{100sv}{27} - \frac{1000}{729}$,

and take $\frac{100sv}{27} = \frac{118v}{27}$, or $s = \frac{59}{50}$, to take away the two last terms on each side of the equation: then the remaining terms will be $v^3 - \frac{10v^2}{3} = s^3v^3 - \frac{10s^2v^2}{3}$, and hence

$v = \frac{10}{3} \times \frac{s^2 - 1}{s^3 - 1} = \frac{10}{3} \times \frac{s + 1}{s^2 + s + 1}$

$$= \frac{10}{3} \div \left(s + \frac{1}{s+1}\right) = \frac{54500}{26793},$$

$$\text{and } n = v - \frac{13}{9} = \frac{15799}{26793},$$

$$\text{and } n^3 + n^2 + n + 1 = \left(sv - \frac{10}{9}\right)^3 = \left(\frac{34540}{26793}\right)^3.$$

Then from the equation $y^3 = \dfrac{1}{n^3 + n^2 + n + 1}$,

$$y = \frac{1}{\sqrt[3]{(n^3 + n^2 + n + 1)}} = \frac{1}{sv - \frac{10}{9}} = \frac{26793}{34540};$$

and $x = ny = \dfrac{15799}{34540}$, which are two fractions that will answer.

THIRD SOLUTION, *by Mr.* CUNLIFFE.

Let the two fractions be denoted by vz and uz; then, by the question,

$v^4z^4 + uz = u^4z^4 + vz$; or $z^3 \times (v^4 - u^4) = v - u$,

and

$$z^3 = \frac{v-u}{v^4 - u^4} = \frac{1}{v^3 + v^2u + vu^2 + u^3} = \frac{1}{(v+u) \times (v^2 + u^2)};$$

and therefore $(v + u) \times (v^2 + u^2)$ must be a cube.

Now the expression $(v + u) \times (v^2 + u^2)$ will obviously be a cube when each of its factors $v + u$ and $v^2 + u^2$ is a cube.

The question is therefore reduced to the finding of two numbers, whose sum, and sum of their squares, shall each be a cube.

A solution to this latter problem may be effected as follows:

Put $v + u = 1$, and $v^2 + u^2 = b^3$; from twice the second of these equations subtract the square of the first, and there will remain

$$v^2 - 2vu + u^2 = 2b^3 - 1,$$

and taking the roots $v - u = \sqrt{(2b^3 - 1)}$;

therefore $2b^3 - 1$ must be a square.

Put $1 + r = b$, then $2b^3 - 1 = 1 + 6r + 6r^2 + 2r^3 =$ a square $= (1 + 3r^2)^2 = 1 + 6r + 9r^2$; whence $r = \frac{3}{2}$, and $b = r + 1 = \frac{5}{2}$.

But the value of b just found is too great, therefore put $b = \frac{5}{2} - s$, whence $2b^3 - 1 = \frac{121}{4} - \frac{75s}{2} + 15s^2 - 2s^3$ which is to be a square. Assume $\frac{11}{2} - \frac{75s}{22}$ for the root of the preceding expression, that is, put $\frac{121}{4} - \frac{75s}{2} + 15s^2 - 2s^3 = \left(\frac{11}{2} - \frac{75s}{22}\right)^2 = \frac{121}{4} - \frac{75s}{2} + \frac{(75)^2s^2}{(22)^2}$; whence $s = \frac{15 \times (22)^2 - (75)^2}{2 \times (22)^2} = \frac{1635}{968}$; and $b = \frac{5}{2} - s = \frac{785}{968}$; and hence $v - u = \sqrt{(2b^3 - 1)} = \frac{5497}{21296}$; whence, and from the equation $v + u = 1$, we get $v = \frac{26793}{42592}$ and $u = \frac{15799}{42592}$.

Again we have the expression $z^3 = \frac{1}{(v + u) \times (v^2 + u^2)} = \frac{1}{v^2 + u^2}$, because $v + u = 1$; wherefore $z^3 = \frac{1}{v^2 + u^2} = \frac{1}{b^3}$; whence $z = \frac{1}{b} = \frac{968}{785}$, and hence $vz = \frac{26793}{34540}$, and $uz = \frac{15799}{34540}$, which are the very fractions found by the other method of solution.

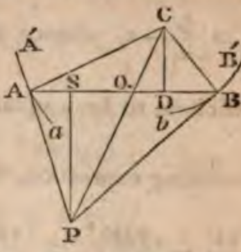
From what has been done, we shall easily find that 53586 and 31598, are two integers, whose sum, and sum of their squares are each a rational cube.

VII. QUESTION 357, *by* QUIDAM.

Given the greatest and least slant sides of an oblique cone, to determine the diameter of the base, that its solid content may be the greatest possible?

FIRST SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

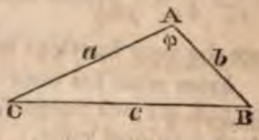
Let ACB represent a section of the cone by a plane passing through its greatest and least slant sides CA, CB. With the centre C and radii CA, CB describe the circular arcs A′Aa, B′Bb and draw the tangents AP, BP meeting each other in P. Draw CD, PS perpendicular to AB; and draw CP cutting AB in Q. It will then manifestly appear, from the scholium to theorem 19 of Simpson's Maxima and Minima of geometrical quantities, that the solid $AB^2 \times CD$ will be the greatest possible when $PS = 2CD$, or $PQ = 2CQ$. The angles CAP, CBP are right angles, therefore a circle whose diameter is CP, will pass through A and B. Then by a known property $CD \times CP = AC \times BC$; and therefore $PB \times PA = PS \times PC = 2CD \times PC = 2AC \times BC$ a given magnitude. Moreover $CP^2 = PB^2 + BC^2 = PA^2 + AC^2$ whence $PB^2 - PA^2 = AC^2 - BC^2$ a given magnitude. Therefore, there is given, the rectangle of the lines PB and PA together with the difference of their squares, from whence the lines themselves become known, and the problem may thereby be readily constructed.



SECOND SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let a and b be the given sides of the cone ACB, φ their included angle, and c the diameter of the base. Because the content of the cone is proportional to $c^2 \times$ by the altitude, that is to $c \times 2\triangle CAB$, therefore

$c \times 2\triangle CAB$ must be a maximum.



But $c = \sqrt{(a^2 + b^2 - 2ab\cos\varphi)}$ and $2\triangle CAB = ab\sin\varphi$; therefore $ab\sin\varphi\sqrt{(a^2 + b^2 - 2ab\cos\varphi)}$ = a maximum, or if $\frac{a^2 + b^2}{2ab}$ be put $= d$, the expression squared, and the constant factor omitted,

$$\sin^2\varphi\,(d - \cos\varphi) \text{ must be a maximum.}$$

In fluxions $2\dot{\varphi}\sin\varphi\cos\varphi\,(d - \cos\varphi) + \dot{\varphi}\sin^3\varphi = 0$,

or $\quad 2d\cos\varphi - 2\cos^2\varphi + \sin^2\varphi = 0$,

and putting $1 - \cos^2\varphi$ for $\sin^2\varphi$, and completing the square,

$$\cos\varphi = \frac{d}{3} \mp \tfrac{1}{3}(3 + d^2),$$

and restoring the value of d,

$$\cos\varphi = \frac{a^2 + b^2}{6ab} \mp \frac{1}{6ab}\sqrt{(14a^2b^2 + a + b^4)},$$

therefore $2ab \cos \phi = \frac{a^2 + b^2}{3} \mp \frac{1}{3}\sqrt{(14a^2b^2 + a^4 + b^4)}$,

and $c = \sqrt{\left\{\frac{2a^2 + 2b^2}{3} \pm \frac{1}{3}\sqrt{(14a^2b^2 + a^4 + b^4)}\right\}}$.

When $a = b$, $\cos \phi = \frac{1}{3} - \frac{2}{3}$,

$$\text{and } c = \sqrt{\left(\frac{4a^2}{3} + \frac{4a^2}{3}\right)} = \sqrt{\frac{8a^2}{3}} = 2a\sqrt{\tfrac{2}{3}}.$$

THIRD SOLUTION, *by* QUIDAM, *the Proposer.*

Let the triangle ACB *(see the figure to the first solution)* represent a section of the cone by a plane passing through its greatest and least slant sides CA, CB and draw CD perpendicular to the base AB. Put AC $= a$, BC $= b$, AD $= x$, and BD $= y$. By a known property, AD^2 — BD^2 = AC^2 — CB^2, that is $x^2 - y^2 = a^2 - b^2$, and taking fluxions $2x\dot{x} - 2y\dot{y} = 0$, or $\dot{y} = \frac{x\dot{x}}{y}$. By the principles of mensuration, the content of the cone is expressed by $\frac{\text{AB}^2 \times \text{CD}}{3} \times \cdot 7854$, which will be a maximum when AB^2 × CD $= (x + y)^2\sqrt{(a^2 - x^2)}$ is so. Putting the fluxion of the last expression $= 0$, $2 \times (\dot{x} + \dot{y}) \times (x + y) \times \sqrt{(a^2 - x^2)} = \frac{x\dot{x} \times (x + y)^2}{\sqrt{(a^2 - x^2)}}$, or $2 \times (\dot{x} + \dot{y}) \times (a^2 - x^2) = x\dot{x} \times (x + y)$.

Exterminating $\dot{y}$ by means of its equal $\frac{x\dot{x}}{y}$, the last equation becomes $2 \times \left(\dot{x} + \frac{x\dot{x}}{y}\right) \times (a^2 - x^2) = x\dot{x}\,(x + y)$; whence $2 \times (a^2 - x^2) = xy$, and $y = \frac{2 \times (a^2 - x^2)}{x}$. And hence again $x^2 - y^2 = x^2 - \frac{4 \times (a^2 - x^2)^2}{x^2} = a^2 - b^2$, which being properly reduced, gives

$$x^2 = \frac{7a^2 + b^2 + \sqrt{(a^4 + 14a^2b^2 + b^4)}}{6}.$$

FOURTH SOLUTION, *by Mr.* J. BAINES, *jun.*

Let ABC (*see the figure to the first solution*) represent the oblique cone, and CD its perpendicular altitude. Put AC $= a$, BC $= b$, and the diameter AB $= x$; then AB : AC + CB :: AC

$-\text{CB} : \text{AD} - \text{DB} = \dfrac{a^2 - b^2}{x}$, and $\dfrac{\text{AB} + \text{AD} - \text{DB}}{2} = \text{AD} = \dfrac{x^2 + a^2 - b^2}{2x}$; also $\sqrt{(\text{AC}^2 - \text{AD}^2)} = \text{CD} = \dfrac{1}{2x}\sqrt{(2a^2x^2 - x^4 - a^4 + 2b^2x^2 + 2a^2b^2 - b^4)}$, the perpendicular altitude: but $\text{AB}^2 \times \text{CD} = \dfrac{x}{2}\sqrt{(2a^2x^2 - x^4 - a^4 + 2b^2x^2 + 2a^2b^2 - b^4)}$ is as the solidity of the cone, and must therefore be a maximum, or its square, $2a^2x^4 - x^6 - a^4x^2 + 2b^2x^4 + 2a^2b^2x^2 - b^4x^2 =$ a maximum. In fluxions,

$$8a^2x^3\dot{x} - 6x^5\dot{x} - 2a^4x\dot{x} + 8b^2x^3\dot{x} + 4a^2b^2x\dot{x} - 2b^4x\dot{x} = 0.$$

and dividing by $2x\dot{x}$, and reducing gives

$$x^4 - \left(\frac{4a^2 + 4b^2}{3}\right) \times x^2 = \frac{2a^2b^2 - a^4 - b^4}{3};$$

therefore $x = \sqrt{\left\{\dfrac{2a^2 + 2b^2 + \sqrt{(a^4 + 14a^2b^2 + b^4)}}{3}\right\}}$ the diameter of cone's base required.

When $b = a$, or when the cone is a right one, the above value of x becomes $a\sqrt{\frac{8}{3}} = 2a\sqrt{\frac{2}{3}}$, as it ought. For when the slant sides are equal, the perpendicular bisects the base; hence $\sqrt{\left(a^2 - \dfrac{x^2}{4}\right)} =$ the perpendicular, and $x^2\sqrt{\left(a^2 - \dfrac{x^2}{4}\right)}$ is as the solidity; therefore $x^2\sqrt{\left(a^2 - \dfrac{x^2}{4}\right)}$, or $4a^2x^4 - x^6 =$ a maximum. In fluxions $16a^2x^3\dot{x} - 6x^5\dot{x} = 0$, from which we obtain $x = 2a\sqrt{\frac{2}{3}}$, the same as before.

VIII. QUESTION 358, *by* QUIDAM.

Find the sum of the infinite series,

$$\frac{1}{3} - \frac{1 \cdot 5}{3 \cdot 7} + \frac{1 \cdot 5 \cdot 9}{3 \cdot 7 \cdot 11} - \frac{1 \cdot 5 \cdot 9 \cdot 13}{3 \cdot 7 \cdot 11 \cdot 15} + \&c.$$

SOLUTION, *by Mr.* WM. WALLACE, *R. M. College.*

To sum the series, let

$$s = \frac{x^3}{3} - \frac{1 \cdot 5}{3 \cdot 7}x^7 + \frac{1 \cdot 5 \cdot 9}{3 \cdot 7 \cdot 11}x^{11} + \frac{1 \cdot 5 \cdot 9 \cdot 13}{3 \cdot 7 \cdot 11 \cdot 15}x^{15} + \&c.$$

Take the fluxions of both sides, and also divide by x^2, and the result is

$$\frac{ds}{x^2} = dx - \frac{1 \cdot 5}{3}dx x^4 + \frac{1 \cdot 5 \cdot 9}{3 \cdot 7}dx x^8 - \frac{1 \cdot 5 \cdot 9 \cdot 13}{3 \cdot 7 \cdot 11}dx x^{12} + \&c.$$

taking now the fluents, we have

$$\int\frac{ds}{x^2} = x - \tfrac{1}{3}x^5 + \frac{1\cdot5}{3\cdot7}\,x^9 - \frac{1\cdot5\cdot9}{3\cdot7\cdot11}\,x^{13} + \&c.$$

The series which forms the second member of this equation is manifestly $= x - sx^2$, therefore,

$$\int\frac{ds}{x^2} = x - sx^2,$$

and hence, taking the fluxions,

$$\frac{ds}{x^2} = dx - x^2ds - 2xsdx\,;$$

This equation reduced to a proper form is

$$ds + \frac{2x^3dx}{1+x^4}s = \frac{x^2dx}{1+x^4}.$$

Multiply both sides by X, some function of x to be presently determined, and it becomes

$$\mathrm{X}ds + s\,\frac{2\mathrm{X}x^3}{1+x^4}\,dx = \frac{\mathrm{X}x^2dx}{1+x^4},$$

the first member will manifestly be an exact integral, if we assume

$$\frac{2\mathrm{X}x^3dx}{1+x^4} = d\mathrm{X},$$

for the equation then becomes

$$\mathrm{X}ds + sd\mathrm{X} = d\,(\mathrm{X}s) = \frac{\mathrm{X}x^2dx}{1+x^4};$$

hence $\mathrm{X}s = \int\frac{\mathrm{X}x^2dx}{1+x^4}$, and $s = \frac{1}{\mathrm{X}}\int\frac{\mathrm{X}x^2dx}{1+x^4}$,

and since $\frac{2d\mathrm{X}}{\mathrm{X}} = \frac{4x^3dx}{1+x^4}$,

therefore $\mathrm{X}^2 = 1 + x^4$, and $\mathrm{X} = \sqrt{(1+x^4)}\,;$

thus we have $s = \frac{1}{\sqrt{(1+x^4)}}\cdot\int\frac{x^2dx}{\sqrt{(1+x^4)}}$,

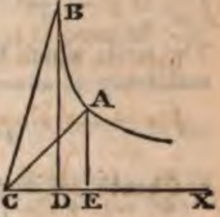
and putting s' for the sum of the proposed series, that is for the value of s when $x = 1$,

$$s' = \frac{1}{\sqrt{2}} \times \int\frac{x^2dx}{\sqrt{(1+x^4)}},$$

the fluent to be taken between $x = 0$, and $x = 1$.

To exhibit the general value of this fluent geometrically, let BA be an equilateral hyperbola, of which the semiconjugate axis =

$\sqrt{2}$, and CX one of its asymptotes; let CE $= x$ and AE $= y$ be the co-ordinates of A any point in the curve, and CD the abscissa corresponding to a given point B. Join CA, CB. Put CA $= r$, and the hyperbolic arc BA $=$ H.



By the nature of the curve, $y = \frac{1}{x}$

therefore, $r = \frac{\sqrt{(1+x^4)}}{x}$, and

$$dr = \frac{2x^2 dx}{\sqrt{(1+x^4)}} - \frac{dx\sqrt{(1+x^4)}}{x^2} = \frac{x^2 dx}{\sqrt{(1+x^4)}} - \frac{dx}{x^2\sqrt{(1+x^4)}},$$

$$d\text{H} = \sqrt{(dx^2+dy^2)} = \frac{dx\sqrt{(1+x^4)}}{x^2} = \frac{x^2 dx}{\sqrt{(1+x^4)}} + \frac{dx}{x^2\sqrt{(1+x^4)}},$$

$$dr + d\text{H} = \frac{2x^2 dx}{\sqrt{(1+x^4)}},\text{ and hence}$$

$$\int \frac{x^2 dx}{\sqrt{(1+x^4)}} = \tfrac{1}{2}(r + \text{H}) + c;$$

Now when $x =$ CD, then $r =$ CB and H $= 0$, therefore the fluent generated between $x =$ CD and $x =$ CE is

$$\tfrac{1}{2}\left\{\text{CA} - (\text{CB} - arc\ \text{BA})\right\}.$$

This is true, whatever be the magnitudes of CD and CE. Suppose now CD $= 0$, and CE $= 1$; then CA is the semi-axis, and CB — *arc* BA becomes the *limit* of the excess of a semidiameter of the hyperbola above the arc intercepted between that diameter and the axis. This is a finite quantity (Simpson's Fluxions, page 509, Ed. 1750) and is expressible by quadrantal arcs of an ellipse and a circle (Landen's Mathematical Memoirs, vol. 1. page 33*). Its numerical value is ·5990701173 × semi-axis, (Landen's Memoirs, vol. 1. Appendix, page 25†).

Hence, the sum of the proposed series

$$s' = \frac{1}{2\sqrt{2}}\left\{\sqrt{2} - \cdot 5990701173 \times \sqrt{2}\right\}$$
$$= \cdot 2004649413.$$

Note. The investigation of the general expression for an hyperbolic arc in terms of two elliptic arcs has been given in quest. 99 of the new series of the *Repository*.

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* The Memoir containing this discovery was given in the first series of the Mathematical Repository, vol. 2, page 3.

† See Mathematical Repository (first series) vol. 1. page 178.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

At page 124, Cor. 3, Prop. 2, Simpson's Dissertations, we have the following expression, viz.

$$\int(a+cz^n)^m \times dz^{en-1}\dot{z} = (a+cz^n)^{m+1} \times \frac{dz^{en}}{ena} \times$$

$$\left\{1-\frac{(t+1)cz^n}{(e+1)a}+\frac{(t+1).(t+2)c^2z^{2n}}{(e+1).(e+2)a^2}-\frac{(t+1).(t+2).(t+3)c^3z^{3n}}{(e+1).(e+2).(e+3)a^3}\right.$$

$\left.+\ \&c.\right\}$; dividing both sides by $(a+cz^n)^{m+1} \times \frac{dz^{en}}{ena}$, we get

$$\frac{ena}{z^{en} \times (a+cz^n)^{m+1}} \times \int(a+cz^n)^m \times z^{en-1}\dot{z} =$$

$$1-\frac{(t+1)cz^n}{(e+1)a}+\frac{(t+1).(t+2)c^2z^{2n}}{(e+1).(e+2)a^2}-\frac{(t+1).(t+2).(t+3)c^3z^{3n}}{(e+1).(e+2).(e+3)a^3}+\&c.$$

Taking a and c each $=1$, and $n=4$, the last expression becomes

$$\frac{4e}{z^{4e} \times (1+z^4)^{m+1}} \times \int(1+z^4)^m \times z^{4e-1}\dot{z} =$$

$$1-\frac{(t+1)z^4}{e+1}+\frac{(t+1).(t+2)z^8}{(e+1).(e+2)}-\frac{(t+1).(t+2).(t+3)z^{12}}{(e+1).(e+2).(e+3)}+\&c.$$

dividing by 3, $\frac{\frac{4}{3}e}{z^{4e} \times (1+z^4)^{m+1}} \times \int(1+z^4)^m \times z^{4e-1}\dot{z} =$

$$\tfrac{1}{3} \times \left\{1-\frac{(t+1)z^4}{e+1}+\frac{(t+1).(t+2)z^8}{(e+1).(e+2)}-\frac{(t+1).(t+2).(t+3)z^{12}}{(e+1).(e+2).(e+3)}+\&c.\right\}$$

Now $\frac{1}{3}-\frac{1.5z^4}{3.7}+\frac{1.5.9z^8}{3.7.11}-\frac{1.5.9.13z^{12}}{3.7.11.15}+\&c.$ is obviously

$$=\tfrac{1}{3} \times \left\{1-\frac{\frac{1}{4}+1}{\frac{3}{4}+1}z^4+\frac{(\frac{1}{4}+1).(\frac{1}{4}+2)}{(\frac{3}{4}+1).(\frac{3}{4}+2)}z^8-\frac{(\frac{1}{4}+1).(\frac{1}{4}+2).(\frac{1}{4}+3)}{(\frac{3}{4}+1).(\frac{3}{4}+2).(\frac{3}{4}+3)}\right.$$

$\left.z^{12}+\&c.\right\}$ and comparing this with the expression just deduced, we shall have $e=\frac{3}{4}$, $t=e+m=\frac{1}{4}$, whence $m=-\frac{1}{2}$: and hence that expression becomes $\frac{1}{z^3\sqrt{(1+z^4)}} \times \int\frac{z^2\dot{z}}{\sqrt{(1+z^4)}}=\frac{1}{3} \times$

$$\left\{1-\frac{\frac{1}{4}+1}{\frac{3}{4}+1}z^4+\frac{(\frac{1}{4}+1).(\frac{1}{4}+2)}{(\frac{3}{4}+1).(\frac{3}{4}+2)}z^8-\&c.\right\}$$

or $\frac{1}{\sqrt{(1+z^4)}} \times \int\frac{z^2\dot{z}}{\sqrt{(1+z^4)}}=\frac{1}{3}z^3-\frac{1.5}{3.7}z^7+\frac{1.5.9}{3.7.11}z^{11}-\&c.$

Taking $z = 1$, the last expression becomes $\frac{1}{\sqrt{2}} \times \int \frac{z^2\dot{z}}{\sqrt{(1+z^4)}}$, generated whilst z from 0 becomes 1,

$$= \tfrac{1}{3} - \frac{1.5}{3.7} + \frac{1.5.9}{3.7.11} - \frac{1.5.9.13}{3.7.11.15} + \&c.$$

At page 149, part 10, sect. 2, Landen's Lucubrations, we have

$$\sqrt{2} - \frac{e - \sqrt{(e^2 - 2f)}}{\sqrt{2}} = \frac{2 + \sqrt{(e^2 - 2f)} - e}{\sqrt{2}} = \int \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{(1+x^2)}}$$

generated whilst x from 0 becomes 1, where $e = 1{\cdot}91009889$ is $\frac{1}{4}$ of the periphery of an ellipse, whose semi-axes are $\sqrt{2}$ and 1: and $f = 1{\cdot}5707963268$ is the quadrantal arc of a circle, radius 1.

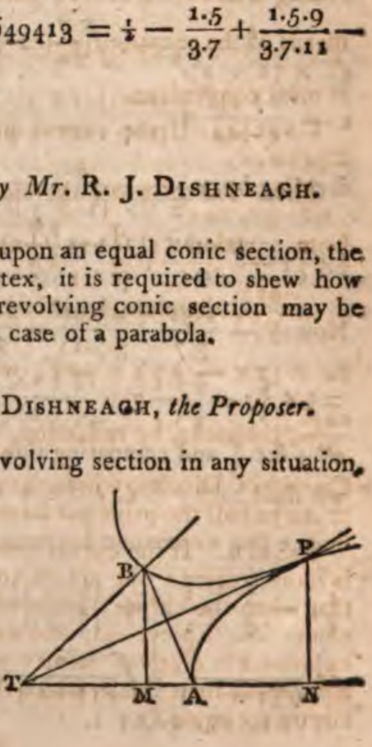
Writing z^2 for x and dividing by 2, we shall have $\frac{2 + \sqrt{(e^2 - 2f)} - e}{2\sqrt{2}} = \int \frac{z^2\dot{z}}{\sqrt{(1+z^4)}}$, generated whilst z from 0 becomes 1, and consequently $\frac{1}{\sqrt{2}} \times \int \frac{z^2\dot{z}}{\sqrt{(1+z^4)}} = \frac{2 + \sqrt{(e^2 - 2f)} - e}{4} = {\cdot}2004649413 = \frac{1}{3} - \frac{1.5}{3.7} + \frac{1.5.9}{3.7.11} - \frac{1.5.9.13}{3.7.11.15} +$ &c. ad infinitum.

IX. QUESTION 359, *by Mr.* R. J. DISHNEAGH.

If any conic section revolve upon an equal conic section, the motion beginning from the vertex, it is required to shew how the locus of the vertex of the revolving conic section may be determined: and apply it to the case of a parabola.

SOLUTION, *by Mr.* R. J. DISHNEAGH, *the Proposer.*

Let B be the vertex of the revolving section in any situation, then by the nature of the revolution the arc BP = arc AP, and, therefore, since they are equal portions of equal curves, they will have a common tangent PT, and therefore T is a point in the axis of the revolving curve, and



therefore TB is in the direction of the axis and = TA: and also the angle BTP = PTN.

Now, let AM $= x$, BM $= y$, $\angle$ ATP $= \varphi$, and x', y' corresponding co-ordinates of the original curve.

Now $\frac{\text{AM}}{\text{BM}}$ = tan. ABM = cot BAM = tan PTN,

therefore $\frac{x}{y} = \tan\varphi =$, therefore, $\frac{\text{PN}}{\text{TN}}$..................(1)

But $y =$ BM = TM × tan BTM = TM × tan 2φ

$$= \text{TM} \times \frac{2\tan\varphi}{1 - \tan^2\varphi} = \text{TM} \times \frac{2\frac{x}{y}}{1 - \frac{x^2}{y^2}} = \text{TM} \times \frac{2xy}{y^2 - x^2},$$

therefore $\frac{y^2 - x^2}{2x}$ = TM = TN — AN — AM = TN — AN — x;

therefore $y^2 + x^2 = 2x \times$ (TN — AN)..................(2)

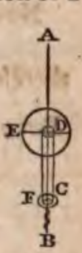
But TN — AN can be determined in terms of x and y by means of the equation $\frac{\text{PN}}{\text{TN}} = \frac{x}{y}$ or by its equivalent $\frac{\dot{y}'}{\dot{x}} = \frac{x}{y}$ as is most convenient.

CASE 1. If the curves are parabolas whose equation is $y'^2 = 4ax'$.

Then TN = 2AN, therefore TN — AN = AN = x'. But from equation (1) $\frac{x}{y} = \frac{\text{PN}}{\text{TN}} = \frac{\sqrt{(4ax')}}{2x'} = \sqrt{\frac{a}{x'}}$; therefore $x' = \frac{ay^2}{x^2}$, therefore, by substitution, in the equation $y^2 + x^2 = 2x \times (\text{TN} - \text{AN}) = 2xx'$, we have $y^2 + x^2 = 2x \times \frac{ay^2}{x^2} = \frac{2ay^2}{x}$, therefore, by reduction, $y^2 = \frac{x^3}{2a - x}$, the equation to the Cissoid of Diocles, where the diameter of the generating circle $= 2a$ or half the principal latus rectum of the parabola.

CASE 2. If the curves were ellipses or circles, the solution is more tedious: but not more difficult. It is easy to perceive that when the ellipse transforms into a circle, that the curve becomes the Epicycloid, and its equation in this case can be expressed in *algebraic* terms: which is not the case in any other Epicycloid than this in which the radii of the fixed and moveable circles are equal.

X. QUESTION 360, *by the Rev. Mr.* L. Evans, *of the Royal Military Academy, Woolwich.*



In the annexed figure, which represents a vertical section of a pendulum, invented by Mr. Adam Reid, of Woolwich, for which he received a premium of the Society of Arts, Adelphi, AB is a steel rod, CD is a perforated cylinder of zinc, supporting the bob E at its centre and resting on the regulating nut F turned by a screw at the lower end of the rod AB; so that as the steel rod contracts or expands, the zinc cylinder will do so likewise, keeping the bob always at the same distance from the point of suspension. Now, supposing the expansion of steel-rod to zinc be in the proportion of 1144·7 to 2942, and that the pendulum is intended to vibrate seconds; what must be the length of the cylinder of zinc to effect the purpose of compensating for heat and cold, not taking into consideration the quantity of matter in the metals?

Solution, *by the Rev. Mr.* L. Evans, *the Proposer.*

Let $l = 39{\cdot}2$ the length of the royal pendulum,
$n = 2942$, $s = 1144{\cdot}7$, and
$x =$ length of the required cylinder of zinc; then $l + x$ will be the whole length of the steel bar, and by the nature of the question $n : s :: l + x : x$; or $n - s : s :: l : x = \frac{ls}{n - s} =$

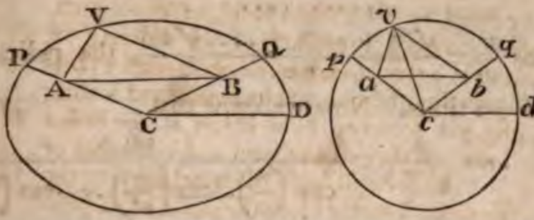
$$\frac{1144{\cdot}7 \times 39{\cdot}2}{2942 - 1144{\cdot}7} = 24.96647 \text{ inches the length sought.}$$

XI. QUESTION 361, *by* G. V.

Let CAP, CBQ be any two given semi-diameters of an ellipse; from any point V in the curve let VA be drawn parallel to a line given by position, to meet one of them in A; there is another line given by position, to which, if VB be drawn parallel, so as to meet the other diameter in B, the line joining A and B shall have a given ratio to CD, the semi-diameter parallel to AB.

Solution, *by Mr.* W. Wallace, *R. M. College.*

Consider the ellipse as the section of a cylinder, and conceive the curve and the lines CP, CQ, CD, VA, VB, AB to be orthographically projected, on a plane perpendicular to the axis of the cylinder, the ellipse will be projected into a circle *pvq*, the semi-diameters CP, CQ, CD will be projected into radii *cp*, *cq*, *cd*, and the lines VA, VB, AB into corresponding lines *va*, *vb*, *ab*. By the nature of the projection, a line which makes a given angle with a given line on a given plane is projected into a line which makes also a given angle with a given line in the plane of projection, therefore, since by hypothesis the lines, VA, VB make given angles with CP, CQ, the lines *va*, *vb* will also make given angles with *cp*, *cq*. Moreover, because the projections of parallel lines are parallel and have the same ratio to each other as the lines themselves, and by hypothesis, AB and CD are parallel and have a given ratio, therefore *ab* and *cd* will also be parallel and have a given ratio: thus it appears that if the proposition in question hold true of the circle, it will also hold true of the ellipse.



Now the proposition will evidently be true in the circle, when the angles *vac*, *vbc* are the supplements of each other, for then, the points *v*, *a*, *c*, *b* are in the circumference of a circle, and if *vc* be drawn, the lines *vc*, *ab* will subtend given angles, and therefore will have a given ratio; hence, in the circle, corresponding to any given angle *vac*, there is always another given angle *vab* such, that *ab* has to *cd* a given ratio, and therefore the same must also be true in the ellipse.

XII. QUESTION 362, *by* J. F. W. H.

To express the ratio (2π) of the circumference of a circle to radius in a formula involving only powers and roots of 2.

SOLUTION, *by the Proposer.*

The known formula of $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{1 + \cos A}{2}}$, gives the following equations

$$\cos\frac{\pi}{2^2} = \frac{\surd 2}{2}$$

$$\cos\frac{\pi}{2^3} = \frac{\surd(2 + \surd 2)}{2}$$

. . . .

$\cos\frac{\pi}{2^i} = \frac{\surd(2 + \surd(2 + \ldots.\surd 2))}{2}$, the radical $\surd$ occurring $i - 1$ times. Now we know that

$$2^{i-1} . \sin\frac{A}{2^{i-1}} = \frac{\sin A}{\cos\left(\frac{A}{2}\right) . \cos\left(\frac{A}{2^2}\right) \ldots . \cos\left(\frac{A}{2^{i-1}}\right)}$$

and consequently

$$2^{i-1} . \sin\frac{\pi}{2^i} = \frac{1}{\cos\left(\frac{\pi}{2^2}\right) . \cos\left(\frac{\pi}{2^3}\right) \ldots . \cos\left(\frac{\pi}{2^i}\right)}.$$

Let now $i = \infty$, and $\sin\frac{\pi}{2^i} = \frac{\pi}{2^i}$, then $\pi = 2 . \frac{1}{\cos\frac{\pi}{2^2}} . \frac{1}{\cos\frac{\pi}{2^3}}$. &c.

and,

$$\pi = 2^2 . \left\{\left(\frac{2}{\surd 2}\right) . \left(\frac{2}{\surd(2 + \surd 2)}\right) . \left(\frac{2}{\surd(2 + \surd(2 + \surd 2))}\right) . \&c. \right\}.$$

Q. E. I.

XIII. QUESTION 363, *by* J. F. W. H.

If the series $\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} -$ &c. be denoted ${}^2\mathrm{L}(1 + x)$, it is required to shew that the following equation holds,

$${}^2\mathrm{L}(x) + {}^2\mathrm{L}\left(\frac{1}{x}\right) = \frac{(\log x)^2}{1 . 2}.$$

Solution, *by the Proposer.*

It is evident, that ${}^2\mathrm{L}(1 + x) = \int\frac{dx}{x}\left(\frac{x}{1} - \frac{x^2}{2} + \&c.\right) = \int\frac{dx}{x} . \log(1 + x)$, and for x writing $x - 1$, we find

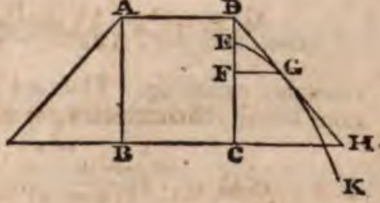
$^2\mathrm{L}(x) = \int \frac{dx}{x-1} \cdot \log x$. In this for x put $\frac{1}{x}$, and we have $^2\mathrm{L}\left(\frac{1}{x}\right) = \int \frac{-dx}{x(x-1)} \cdot \log x$.

whose sum is, $^2\mathrm{L}(x) + {}^2\mathrm{L}\left(\frac{1}{x}\right) = \int \frac{dx \cdot \log x}{x-1}\left(1-\frac{1}{x}\right) = \int \frac{dx \log x}{x} = \frac{(\log x)^2}{2} + \mathrm{C}$. Now $^2\mathrm{L}(1) = {}^2\mathrm{L}(1+0)$. Let then $x=1$, and we find $\mathrm{C} = 2 \cdot {}^2\mathrm{L}(1) = 0$, thus, $^2\mathrm{L}(x) + {}^2\mathrm{L}\left(\frac{1}{x}\right) = \frac{(\log x)^2}{1 \cdot 2}$.
It is remarkable that this, undoubtedly the most elegant property of the second order of logarithmic transcendents, although implied in some operations in Spence's work on the subject, has not been thought deserving of a place in his table of the properties of the function $^2\mathrm{L}$, an omission the more to be regretted as it is a property of very great use in the numerical evaluation of $^2\mathrm{L}\ (x)$ when x is a high number. We should observe that it may be derived by a proper combination of the equations (A), (B), (C), of his "table of properties, &c." page 12, Log. Transc.

XIV. QUESTION 364, *by* J. F. W. H.

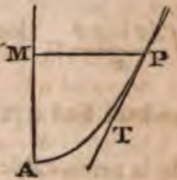
If a cylindrical vessel placed vertically and kept full of water, be bored in innumerable points, the issuing fluid will be bounded by the surface of a conical frustum. Required a proof.

SOLUTION, *by the Proposer.*



ABCD being the vessel and E one of the holes, let EGK be the parabola described by the issuing fluid. Take EF = ED and erect the ordinate FG which, (by the theory of spouting fluids) will be equal to FD, and the angle GDF, 45°. Again, since FD = 2EF, DG is a tangent at G. In the same way it may be shewn that the line DGH, making an angle 45° with the side of the vessel, touches any other of the parabolic jets. Let the whole figure revolve round its axis, and HD will generate the frustum of a conic surface, the boundary of the issuing fluid.

XV. QUESTION 365, *by* J. F. W. H.



Let AM be the axis, MP an ordinate, and PT the tangent of a common catenaria; AM, and the tangent of MPT will be certain functions of MP, which we will denote by C (MP) and S (MP); then supposing the length of the portion of chain equal in weight to the tension at A, to be unity, the functions C and S will have the following properties,

1 .... $C(A)^2 - S(A)^2 = 1$
2 .... $S(A+B) = S(A) \cdot C(B) + (CA) : S(B)$
3 .... $C(A+B) = C(A) \cdot C(B) + S(A) \cdot S(B)$
4 .... $C(2A) = 2 \cdot C(A)^2 - 1$.

SOLUTION, *by the Proposer.*

The well known differential equation $dx = z \cdot dy$, gives by making $d^2y = 0$,

$$\frac{d^2x}{dy^2} = \sqrt{(1 + \frac{dx}{dy})};$$

or if $\frac{dx}{dy} = p$; $y = \int \frac{dp}{\sqrt{(1+p^2)}} = \log \left\{p + \sqrt{(1+p^2)}\right\}$;

consequently if $\log E = 1$, $E^y = p + \sqrt{(1+p^2)}$,

$\frac{E^y - E^{-y}}{2} = p = \frac{dx}{dy} = \tan MPT = S(y)$ by the definition;

also $x = \int dy \cdot S(y) = \frac{E^y + E^{-y}}{2} = C(y)$; the arbitrary constants vanishing. Thus we have for the forms of the functions whose characteristics are C and S,

$$C(A) = \frac{E^A + E^{-A}}{2}; \quad S(A) = \frac{E^A - E^{-A}}{2}$$

and it is easy to see by mere substitution that these functions are possessed of the properties above proposed.

XVI. QUESTION 366, *by* J. F. W. H.

If peas be taken at random out of a bag; there is a greater probability of taking an odd than an even number. A proof is required.

SOLUTION, *by the Proposer.*

Let the number in the bag be x; then the number of different ways of taking out 1, 3, 5, ... and 2, 4, 6, ... respectively will be

$$\frac{x}{1}, \quad \frac{x(x-1)(x-2)}{1.2.3}, \quad \frac{x\ldots\ldots(x-4)}{1.2.3.4.5}, \ldots\ldots$$

$$\text{and,} \quad \frac{x(x-1)}{1.2}, \quad \frac{x(x-1)(x-2)(x-3)}{1.2.3.4}, \ldots\ldots$$

the sum of all amounts to

$$\frac{x}{1} + \frac{x(x-1)}{1.2} + \&c. = 2^x - 1.$$

The sum of the ways in which an odd number may be extracted is

$$\frac{x}{1} + \frac{x(x-1)(x-2)}{1.2.3} + \&c. = 2^{x-1}$$

and the probability of an odd number $= \dfrac{2^{x-1}}{2^x - 1}$.

Again, the sum of the ways of taking an even number, is

$$\frac{x(x-1)}{1.2} + \frac{x\ldots\ldots(x-3)}{1\ldots4} + \&c. = 2^{x-1} - 1$$

and the probability $\dfrac{2^{x-1}}{2^x-1} - \dfrac{1}{2^x-1}$, which gives an excess of probability on the side of the odd numbers amounting to $\dfrac{1}{2^x-1}$.

XVII. QUESTION 367, *by* J. F. W. H.

Let $\varepsilon = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c.$ It is required to shew that the definite integral $\frac{1}{\pi}\int d\theta \,.\, \varepsilon^{2\cos\theta}$ between the limits $\theta = 0$, $\theta = \pi = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - \&c.\right\}$, is equal to the series $1^2 + \left(\frac{1}{1}\right)^2 + \left(\frac{1}{1.2}\right)^2 + \left(\frac{1}{1.2.3}\right)^2 + \&c.$

Solution, *by the Proposer.*

Let $f(x) = A_0 + A_1 . x + A_2 . x^2 + \&c.$ For x write successively $\varepsilon^{\theta\sqrt{-1}}$ and $\varepsilon^{-\theta\sqrt{-1}}$, and the results being multiplied together, it is easy to perceive that the result will take the form

$$f\left\{\varepsilon^{\theta\sqrt{-1}}\right\} . f\left\{\varepsilon^{-\theta\sqrt{-1}}\right\} = (A_0)^2 + (A_1)^2 + (A_2)^2 + \&c.$$
$$+ S\left\{B_i\left(\varepsilon^{i\theta\sqrt{-1}} + \varepsilon^{-i\theta\sqrt{-1}}\right)\right\},$$

where S placed before any function of i is used to denote the series whose general term is that function, i having every integer value from $-\infty$ to $+\infty$.

Let this equation, multiplied by $d\theta$, be integrated from $\theta = 0$, to $\theta = \pi$, and since $\int d\theta . S\left\{B_i\left(\varepsilon^{i\theta\sqrt{-1}} + \varepsilon^{-i\theta\sqrt{-1}}\right)\right\} = \int S\left\{B_i . 2\cos i\theta . d\theta\right\} = 0$ between those limits, we find,

$$\frac{1}{\pi}\int d\theta . f\left\{\varepsilon^{\theta\sqrt{-1}}\right\} . f\left\{\varepsilon^{-\theta\sqrt{-1}}\right\} = (A_0)^2 + (A_1)^2 + (A_2)^2 + \&c.$$

Let $f(x) = \varepsilon^x$, and $\therefore A_0 = 1$, $A_1 = \frac{1}{1}$, $A_2 = \frac{1}{1.2}$ &c.

The equation then becomes,

$$\frac{1}{\pi}\int d\theta . \varepsilon^{2\cos\theta} = 1^2 + \left(\frac{1}{1}\right)^2 + \left(\frac{1}{1.2}\right)^2 + \&c.$$

Another Solution.

Let us consider the more general definite integral $\frac{1}{\pi}\int d\theta . \varepsilon^{2\lambda . \cos\theta}$, λ being any number, positive or negative, integral, fractional or imaginary. By putting $\cos\theta = x$, this becomes $\frac{1}{\pi}\int\frac{dx . \varepsilon^{2\lambda x}}{\sqrt{(1-x^2)}}$, the limits of the integral being -1 and $+1$. Now by the expansion of $\varepsilon^{2\lambda x}$ this becomes,

$\frac{1}{\pi}\int\frac{dx}{\sqrt{(1-x^2)}}\left\{1 + \frac{2\lambda x}{1} + \frac{(2\lambda)^2 x^2}{1.2} + \&c.\right\}$. Now it is sufficiently well known that,

$$\int\frac{x^{2n}dx}{\sqrt{(1-x^2)}}=\pi\,.\,\frac{1.3.5\ \ldots(2n-1)}{2.4.6\ldots.(2n)}=\frac{\pi}{2^n}\,.\,\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots.n}$$

$\int\frac{x^{2n-1}dx}{\sqrt{(1-x^2)}}=0.$ Substituting these values, we find

$$\frac{1}{\pi}\int d\theta\,.\,\varepsilon^{2\lambda\,.\,\cos\theta}=1+\frac{(2\lambda)^2}{1.2}\,.\,\frac{1}{2}+\frac{(2\lambda)^4}{1.2.3.4}\,.\,\frac{1.3}{2^2.1.2}+\&c.$$

$$=1+\frac{\lambda^2}{1^2}+\frac{\lambda^4}{(1\,.\,2)^2}+\frac{\lambda^6}{(1\,.\,2\,.\,3)^2}+\&c.$$

Thus, the series $1^2+\frac{x}{1^2}+\frac{x^2}{(1\,.\,2)^2}+\frac{x^3}{(1\,.\,2\,.\,3)^2}+$ &c. is represented by the definite integral $\frac{1}{\pi}\int d\theta\,.\,\varepsilon^{2\sqrt{(x)}\,.\,\cos\theta}$, taken from $\theta=0$, to $\theta=\pi$. Let $x=1$, and the result is the proposed theorem. If $x=-1$, we have this remarkable result

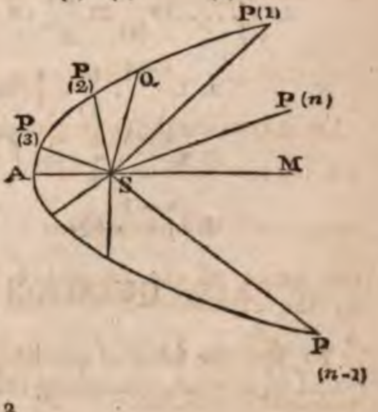
$\frac{1}{\pi}\int d\theta\,.\,\varepsilon^{2\,.\,\cos\theta\sqrt{(-1)}}=1^2-\frac{1}{1^2}+\left(\frac{1}{1\,.\,2}\right)^2-\&c.=\frac{1}{\pi}\,.\,\Big\{\int d\theta\,.$ $\cos\,.\,2\cos\theta+\sqrt{(-1)}\,.\,\sin 2\cos\theta\Big\}$, now since all the elements of $\int d\theta\,.\,\sin\,(2\cos\theta)$ are real, the whole integral is so, and of course must equal 0, thus

$$\int d\theta\,.\,\sin\,(2\,.\,\cos\theta)=0.$$

$$\int d\theta\,.\,\cos\,(2\,.\,\cos\theta)=\pi\,.\,\Big\{1^2-\left(\frac{1}{1}\right)^2+\left(\frac{1}{1\,.\,2}\right)^2-\&c.\Big\}.$$

XVIII. QUESTION 368, *by* J. F. W. H.

Let S be the focus of a parabola $P_{(1)}\,P_{(2)}\ldots.P_{(n)}$ whose vertex is A, axis AM, and latus rectum L. Draw any line $SP_{(1)}$ and make n angles $P_{(1)}SP_{(2)}$, $P_{(2)}SP_{(3)}\ldots.P_{(n)}SP_{(1)}$ round S, all equal to each other. Draw SQ so that the angle MSQ $=n$ times the angle $MSP_{(1)}$. It is required to prove that $SP_{(1)}\,.\,SP_{(2)}\ldots.\,SP_{(n)}=L^{n-1}\,.\,SQ.$



SOLUTION, *by the Proposer.*

Let the angle $\text{MSP}_{(1)}$ be θ, then by the property of the parabola $\text{SP}_{(1)} = \frac{\text{L}}{4} \cdot \left(\sin\frac{\theta}{2}\right)^{-2}$, and (since $\text{P}_{(1)}\,\text{SP}_{(2)} = \&\text{c.} = \frac{2\pi}{n}$, where $\pi = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - \&\text{c.}\right\}$).

$$\text{SP}_{(2)} = \frac{\text{L}}{4} \cdot \left(\sin\left\{\frac{\theta}{2} + \frac{\pi}{n}\right\}\right)^{-2}$$

$$\text{SP}_{(3)} = \frac{\text{L}}{4} \cdot \left(\sin\left\{\frac{\theta}{2} + \frac{2\pi}{n}\right\}\right)^{-2}$$

$$\cdots\cdots\cdots\cdots$$

$$\text{SP}_{(n)} = \frac{\text{L}}{4} \cdot \left(\sin\left\{\frac{\theta}{2} + \frac{(n-1)\pi}{n}\right\}\right)^{-2}$$

consequently, multiplying all together,

$$\text{SP}_{(1)} \ldots \text{SP}_{(n)} =$$

$$\frac{\text{L}^n}{2^{2n}} \cdot \left\{\sin\left(\frac{\theta}{2}\right) \cdot \sin\left(\frac{\theta}{2} + \frac{\pi}{n}\right) \ldots\ldots \sin\left(\frac{\theta}{2} + \frac{(n-1)\pi}{n}\right)\right\}^{-2}.$$

Now it is known that the product

$$\sin(\text{A}) \cdot \sin\left(\text{A} + \frac{\pi}{n}\right) \ldots\ldots \sin\left(\text{A} + \frac{(n-1)\pi}{n}\right)$$

is represented by $2^{1-n} \cdot \sin n\text{A}$, and thus we obtain

$$\text{SP}_{(1)} \ldots\ldots \text{SP}_{(n)} = \frac{\text{L}^n}{2^{2n}} \cdot 2^{-2+2n} \cdot \left\{\sin\left(\frac{n\theta}{2}\right)\right\}^{-2}$$

$$= \text{L}^{n-1} \cdot \frac{\text{L}}{4} \cdot \left\{\sin\left(\frac{n\theta}{2}\right)\right\}^{-2}$$

$$= \text{L}^{n-1} \cdot \frac{\text{L}}{4} \cdot \sin\left(\frac{\text{MSQ}}{2}\right)^{-2}$$

$$= \text{L}^{n-1} \cdot \text{SQ}. \qquad \text{Q. E. D.}$$

XIX. QUESTION 369, *by* J. F. W. H.

To find the form of equilibrium of an arch built from one planet to another, conceiving each particle animated with a force directed towards the sun, and varying inversely as the square of the distance from its centre.

SOLUTION, *by the Proposer.*

Assuming the sun's centre as the origin of two rectangular co-ordinates, x, y of any element ds of the arch, let us call t the thrust exerted by that element, in a tangential direction, at the extremity corresponding to the abscissa x, and of course, $t + dt$ will be the thrust exerted at the corresponding extremity of the consecutive element, which therefore will exert by its re-action, a force $-(t + dt)$ on that extremity of ds which answers to the abscissa $x + dx$, and in the direction of a tangent at that extremity. The forces reduced to the directions of the co-ordinates x and y, become, respectively

$$t \cdot \frac{dx}{ds}; \; -(t + dt) \cdot \frac{d(x + dx)}{d(s + ds)} = -\left\{t \cdot \frac{dx}{ds} + d\left(t \cdot \frac{dx}{ds}\right)\right\}$$

$$t \cdot \frac{dy}{ds}; \; -(t + dt) \cdot \frac{d(y + dy)}{d(s + ds)} = -\left\{t \cdot \frac{dy}{ds} + d\left(t \cdot \frac{dy}{ds}\right)\right\}.$$

Let us, to embrace the question with greater generality, suppose θ to represent the density of the element ds, and $\phi(\sqrt{x^2 + y^2}) = \phi(r)$, the attractive force on a particle at the distance r, and we shall have for the two attractions on ds in the directions x, y,

$$\theta \cdot \phi(r) \cdot \frac{x}{r}\, ds, \text{ and } \theta \cdot \phi(r) \cdot \frac{y}{r}\, ds.$$

The total forces then which animate ds, and which, in virtue of the equilibrium must vanish, are

$$\theta \cdot \phi(r) \cdot \frac{x}{r}\, ds + t \cdot \frac{dx}{ds} - \left\{t \cdot \frac{dx}{ds} + d\left(t \cdot \frac{dx}{ds}\right)\right\}$$

$$\theta \cdot \phi(r) \cdot \frac{y}{r}\, ds + t \cdot \frac{dy}{ds} - \left\{t \cdot \frac{dy}{ds} + d\left(t \cdot \frac{dy}{ds}\right)\right\}$$

whence we find the following equations,

$$(1) \ldots\ldots\ldots\ldots\ldots 0 = \theta \cdot \phi(r) \cdot \frac{x}{r} \cdot ds - d\left(t \frac{dx}{ds}\right)$$

$$(2) \ldots\ldots\ldots\ldots\ldots 0 = \theta \cdot \phi(r) \cdot \frac{y}{r} \cdot ds - d\left(t \frac{dy}{ds}\right)$$

The elimination of t from these conducts to the equation of the curve sought. To accomplish this, multiply (1) by $t \cdot \frac{dx}{ds}$ and (2) by $t \cdot \frac{dy}{ds}$. Their sum is

$$0 = \theta . t . \varphi(r) . \frac{xdx + ydy}{r} - \tfrac{1}{2} d \left\{ t^2 . \frac{dx^2 + dy^2}{ds^2} \right\}$$

or, observing that $xdx + ydy = rdr$, and $dx^2 + dy^2 = ds^2$,

$$0 = t\theta . \varphi(r)\, dr - \tfrac{1}{2} d . t^2, \text{ whence,}$$

$$(3) \ldots\ldots\ldots\ldots\ldots t = \int \theta . \varphi(r) . dr,$$

Again, multiply (1) by y and (2) by x, and let the results be subtracted, thus,

$$0 = y . d\left(t \frac{dx}{ds}\right) - x . d\left(t \frac{dy}{ds}\right)$$

or, supposing $d^2 s = 0$, and reducing

$$0 = \frac{dt}{t} + \frac{d\{ydx - xdy\}}{ydx - xdy}$$

an equation which, integrated with respect to s, gives

$$(4) \ldots\ldots\ldots\ldots\ldots 0 = a ds + t\{ydx - xdy\}.$$

Assume now ψ, so that $x = r . \cos\psi$, $\therefore y = r . \sin\psi$, $ds = \sqrt{(dr^2 + r^2 d\psi^2)}$, $ydx - xdy = -r^2 d\psi$. Thus, equation (4) becomes,

$$a . \sqrt{(dr^2 + r^2 d\psi^2)} = r^2 d\psi . \int \theta\varphi(r) . dr \,; \ldots\ldots\ldots (5).$$

If θ be any assigned function of r, we find from this, as our final equation,

$$\psi + c = a . \int \frac{dr}{r . \sqrt{\{r \int \theta . \varphi(r)\, dr\}^2 - a^2}}.$$

In the case immediately before us, this gives (since $\varphi(r) = \frac{1}{r^2}$, $\theta = 1$), $t = -\frac{1}{r} + const.$ and $const. = 0$, $\therefore t = -\frac{1}{r}$, thus,

$$\psi = \tan \mathrm{A} . \log\left(\frac{r}{a}\right).$$

A and a being two arbitrary constants, which is the well known equation of the logarithmic spiral. In general, the same result will be obtained whenever $\theta = \frac{1}{r^2 . \varphi(r)}$; for instance, if the central force $\propto \frac{1}{r^n}$ and $\theta \propto r^{n-2} \propto s^{n-2}$, in all these cases the tangential thrust is $t = -\frac{1}{r}$.

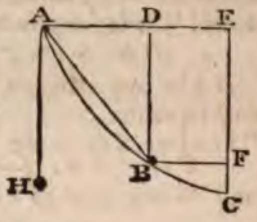
If the force $\propto \frac{1}{r^3}$, and $\theta = 1$, we get $2ar = \sin(\psi + c)$ the equation of a circle passing through the two planets and sun; and $t = -\frac{1}{2r^2}$.

XX. PRIZE QUESTION 370, *by Mr.* LOWRY.

If two given weights be fastened to the ends of a string, which passes over a fixed pulley, and the greater weight descend by the force of gravity, and draw up the other along a curve, which is in the same vertical plane as the pulley: Required the nature of the curve when the time of ascent from one given point to another is the least possible?

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

Let ABC be the required curve in which the body B ascends from C to A in the least time possible when drawn up by the body A which descends vertically.



Let z = the space descended by A in the time B has ascended through the arc CB, and put x = AD, y = DB, ds = the fluxion of the arc CB, v = the velocity of B in the curve, a = EC, c = AC the distance from A to C, $g = 32\frac{1}{6}$ feet the force of gravity.

Then, Art. V. Part II. Vol. III. Repository,

$$v = ds\sqrt{\left\{\frac{2g(\mathrm{A}z - \mathrm{B}(a-y))}{\mathrm{A}dz^2 + \mathrm{B}ds^2}\right\}}.$$

$$\text{Therefore } dt = \frac{ds}{v} = \sqrt{\left\{\frac{\mathrm{A}dz^2 + \mathrm{B}ds^2}{2g(\mathrm{A}z - \mathrm{B}(a-y))}\right\}}$$

$$\text{and } t = \int\sqrt{\left\{\frac{\mathrm{A}dz^2 + \mathrm{B}ds^2}{2g(\mathrm{A}z - \mathrm{B}(a-y))}\right\}}.$$

Put $p = \frac{dy}{dx}$, then because $z = c - \mathrm{AB} = c - \sqrt{(x^2 + y^2)}$,

$$dz = -\frac{xdx + ydy}{\sqrt{(x^2 + y^2)}} = -\frac{dx(x + py)}{\sqrt{(x^2 + y^2)}}$$

$$\text{and } ds = -dx\sqrt{(1 + p^2)}.$$

Therefore substituting these values in the above expression for t, it becomes

$$t = \int dx \sqrt{\left\{\frac{\frac{\mathrm{A}\,(x+py)^2}{x^2+y^2} + \mathrm{B}\,(1+p^2)}{2g\,(\mathrm{AC} - \mathrm{A}\,\sqrt{(x^2+y^2)} - \mathrm{B}\,(a-y))}\right\}}.$$

therefore the fluent of

$$dx \sqrt{\left\{\frac{\frac{\mathrm{A}\,(x+py)^2}{x^2+y^2} + \mathrm{B}\,(1+p^2)}{\mathrm{AC} - \mathrm{A}\sqrt{(x^2+y^2)} - \mathrm{B}(a-y)}\right\}}$$

taken between the limits of $x = 0$ and $x = \mathrm{AE}$ must be a maximum.

$$\text{Put } \mathrm{V} = \sqrt{\left\{\frac{\frac{\mathrm{A}\,(x+py)^2}{x^2+y^2} + \mathrm{B}\,(1+p^2)}{\mathrm{AC} - \mathrm{A}\sqrt{(x^2+y^2)} - \mathrm{B}\,(a-y)}\right\}}$$

then taking fluxions

$$d\mathrm{V} = \mathrm{M}dx + \mathrm{N}dy + \mathrm{P}dp \ldots\ldots\ldots\ldots\ldots (a');$$

and it has been shewn by Euler and by other writers (see Woodhouse on Variations), that when $\int \mathrm{V}dx$ is a maximum or a minimum, V being any function of x, y and p,

$$\mathrm{N} - \frac{d\mathrm{P}}{dx} = 0.$$

Therefore $\mathrm{N}dy = d\mathrm{P}\,\frac{dy}{dx} = p d\mathrm{P}$;

and adding $\mathrm{P}dp$ to each side of the equation,

$$\mathrm{N}dy + \mathrm{P}dp = pd\mathrm{P} + \mathrm{P}dp = d\,(p\mathrm{P});$$

whence by substituting $d\,(p\mathrm{P})$ for $\mathrm{N}dy + \mathrm{P}dp$ in equation a', we have

$$d\mathrm{V} = \mathrm{M}dx + d\,(p\mathrm{P}),$$

and by integration,

$$\mathrm{V} = \int \mathrm{M}dx + p\mathrm{P} + c' \ldots\ldots\ldots\ldots\ldots\ldots (b')$$

c' being the correction.

Whence if values of M and P be found by actually taking the fluxion of

$$\sqrt{\left\{\frac{\frac{\mathrm{A}\,(x+py)^2}{x^2+y^2} + \mathrm{B}\,(1+p^2)}{\mathrm{AC} - \mathrm{A}\sqrt{(x^2+y^2)} - \mathrm{B}\,(a-y)}\right\}}$$

relative to x and p, and be substituted for M and P in formula b' we shall obtain the differential equation of the curve sought: the result however is complex and the equation not easy to integrate.

If we suppose the body B to descend in the curve AC and draw up the body A, we find, in a similar manner,

$$t = \int dx \sqrt{\left\{\frac{B(1+p^2) + \frac{A(x+py^2)}{x^2+y^2}}{2g\,(By - A(x^2+y^2))}\right\}}$$

and in this case when $A = 0$, the question is the same as the common problem of the *brachystochrone*, and the expression for t is then $\int dx \sqrt{\left(\frac{1+p^2}{2gy}\right)}$ as it ought to be.

$$\text{Also } V \text{ is then } = \sqrt{\left(\frac{1+p^2}{y}\right)}$$

$$M = 0,\ P = \frac{p}{\sqrt{\{y(1+p^2)\}}}.$$

Therefore by formula b'

$$V = \frac{p^2}{\sqrt{\{y(1+p^2)\}}} + c';$$

$$\text{that is } \sqrt{\left(\frac{1+p^2}{y}\right)} = \frac{p^2}{\sqrt{\{y(1+p^2)\}}} + c'.$$

and multiplying by $\sqrt{\{y(1+p^2)\}}$

$$1 = c'\sqrt{\{y(1+p^2)\}}$$

$$\text{where } p = \sqrt{\left(\frac{1-c'^2y}{c'^2y}\right)}$$

$$\text{or } dy = \frac{dx}{c'}\sqrt{\left(\frac{1-c'^2y}{y}\right)}$$

the differential equation to the cycloid.

NOTICES.

I. OBITUARY.

LAGRANGE.

The death of this truly great mathematician ought to have been noticed at an earlier period in the Repository, for he closed his mortal career as far back as the 10th of April 1813 in the 78th year of his age.

Our readers are well aware how much this illustrious man has contributed to the improvement of every branch of both pure and mixed mathematical science. Many of his discoveries would singly have conferred on him immortal fame; altogether they form a wreath of glory that will for ever encircle his name, and render it conspicuous in the history of Science.

M. Lagrange was latterly engaged in the publication of a second edition of his *Mécanique Analytique* and the exertion of mind which this labour required is supposed to have brought on the illness which terminated in his death. He had given the first volume to the world, but was cut off while engaged in the second. This volume has since been published; the portion that was unfinished being supplied from the first edition.

M. Delambre has written an Eloge on Lagrange, of which a translation into English has been given in Thomson's Annals of Philosophy, vol. 3. Also in the 10th vol. of the Edinburgh Encyclopedia. His Remains were deposited in the Pantheon, and the following inscription is inscribed upon his Tomb:

Joseph Louis Lagrange,
Sénateur, Comte de L'Empire,
Grand Officier de la Légion d'Honneur,
Grand Croix de L'Ordre Imperial de la Réunion.
Membre de l'Institut, et du Bureau des Longitudes
Né à Turin Département du Po, le XXV Janvier 1736,
Décédé a Paris le X Avril 1813.

A complete list of M. Lagrange's writings is given in the 2nd volume of his Mécanique Analytique. It is composed of two parts, viz. his separate works and memoirs in Academical and other collections. The former are as follows:

Lettre du 23 juin 1754, adressée à Jules-Charles Fagnano, contenant une série pour les différentielles et les intégrales d'un ordre quelconque, correspondante à celle de Newton, pour les puisances (*printed at Turin*).

Additions à l'Algèbre d'Euler.

Mécanique Analytique; 1st edition in 1788; 2nd edition, 1st vol. in 1811; 2nd vol. in 1815.

Théorie des Fonctions Analytiques; 1st edition in 1797; 2nd edition in 1813.

Résolution des Equations Numériques; 1st edit. in 1798; 2nd edit. in 1808.

Leçons sur le Calcul des Fonctions. The first edition made a part of the second edition of the Sèances de l'Ecole Normale, in 1801; this work was also printed in Cahier XII. du Journal de l'Ecole Polytechnique in 1804; and in 1806 the author published a new edition in 8vo. with two additional lessons, which have been inserted in Cahier XIV. du Journal de l'Ecole Polytechnique in 1808.

As his academical memoirs are too numerous to be particularly described in this place, we shall only observe that they are contained in the following works.

Miscellanea Taurinensia.
Mèmoires de l'Académie de Turin.
Mèmoires de l'Académie de Berlin.
Nouveaux Mèmoires de l'Académie de Berlin.
Recueils de l'Académie des Sciences de Paris.
Savans Etrangers.
Institut de France.
Journal de l'Ecole Polytechnique.
Sèances des Ecoles Normales.
Connaissances des Tems. 1814—15.
Une collection de divers ouvrages d'Arithmetique politique, par Lavoisier, Lagrange et autres.

The late Mr. WILLIAM SPENCE, *of Greenock.*

This ingenious mathematician died at Glasgow, on the 22nd of May 1815, in the 37th year of his age. His many private virtues and amiable qualities had endeared him to his friends, and his scientific acquirements and inventive genius had placed him in the highest class of mathematicians in Britain. His essay on the theory of the various orders of Logarithmic Transcendents, published in 1809, was much admired by the few who were able to appreciate its merit, and it gave hopes that science would have derived further benefit from his labours. These we have reason to believe will not be altogether frustrated. His manuscripts were some time ago submitted to Mr. J. F. W. Herschel, of St. John's College, Cambridge, who has selected the most complete for publication. The students of pure mathematics will be gratified to hear that the volume now preparing, and which will be published in the course of the spring by Mr. Underwood, and Messrs. Davis and Dickson, London, contains besides the ingenious Essay on Logarithmic Transcendents, unpublished Tracts in the same class of the science, equally new and elegant. A biographical sketch of the author by his friend Mr. Galt, will be prefixed to the volume.

JAMES GLENIE, Esq. F. R. S. *London and Edinburgh.*

This ingenious mathematician died at his lodgings, at Chelsea, in Nov. 1817, in the 67th year of his age. He was the author of a mathematical theory founded on the doctrine of ratios, which he called the Antecedental Calculus. Also of some papers published in the Philosophical Transactions of London and Edinburgh; a small tract on Gunnery, besides various pamphlets not connected with mathematics.

II. THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON, *for* 1815, *and* 1816.

The mathematical papers contained in the volume for 1815 are, 1. Description of a new instrument for performing mechanically the involution and evolution of numbers, by P. M. Roget, M. D.—2. Three papers containing the depolarization of Light, &c. and two more on the optical properties of Light, by Dr. Brewster.—3. A series of observations of the satellites of the Georgian Planet, by Dr. Herschel.—4. On the dispersive power of the atmosphere, and its effects on astronomical observations, by Mr. Lee.—5. Determination of the North Polar distances and proper motion of thirty fixed stars, by John Pond, Esq. Astronomer Royal, F. R. S.—6. An Essay towards the Calculus of Functions, by C. Babbage, Esq.

The mathematical papers contained in the volume for 1816 are, 1. On the developement of exponential Functions; together with several new theorems relating to finite differences, by John F. W. Herschel, Esq. F. R. S.—2. Direct and expeditious method of calculating the excentric from the mean Anomaly of a Planet, by the Rev. Abram Robertson, D. D. F. R. S.—3. Demonstration of the late Dr. Maskelyne's Formulae, for finding the longitude of a celestial object from its right ascension and declination; and for finding its right ascension and declination from its longitude and latitude, the obliquity of the ecliptic being given in both cases, by the same.—4. An essay towards the Calculus of Functions, part II, by C. Babbage, Esq.—5. A new demonstration of the Binomial Theorem, by Thomas Knight, Esq. 6. On the fluents of Irrational Functions, by E. F. Bromhead, Esq. M. A.

III. INSTITUTE OF FRANCE. Volumes XI, XII, XIII.

The mathematical papers in vol. XI are, 1. On definite integrals, and their application to probabilites, by M. La Place.—2. On certain new optical phenomena, by M. Malus.—3. On the axis of refraction of crystals and organic substances, by the same.—4. On the method of least squares, by M. Legendre.—5. On the attraction of homogeneous ellipsoids, by the same.

The mathematical papers contained in vol. XII are, 1. On the distribution of electricity at the surface of conducting bodies; first and second memoir, by M. Poisson.—2. On a remarkable modification experienced by the rays of light, in their passage through certain diaphanous bodies; and some other new optical phenomena, by M. Arrago—3. On the new relations which exist between the reflection and the polarization of light by crystallized bodies, by M. Biot.

Volume XIII contains 1. On a new kind of oscillation which the particles of light experience in traversing certain crystals, by M. Biot.—2. On a new application of the theory of the oscillations of Light, by the same.—3. On the discovery of a new property in the polarizing forces of certain crystals, by the same.—4. On the physical properties that the particles of light acquire in traversing double refracting crystals, by the same.—5. Result of the meteorological observations made at Clermont-Ferrand, from the month of June 1806 to the end of 1813, by the Baron Ramond.—6. On elastic surfaces, by M. Poisson.

IV. JOURNAL DE L'ECOLE POLYTECHNIQUE.

The Mécanique Philosophique of Prony (which was never finished) formed originally the 7th and 8th Cahiers of this work: it is now left out, and the blank filled up by the Lessons given in the Normal School by Lagrange and Laplace; and a translation of Fermat's Treatise on the Contact of Spheres by Hachette. The new edition of Lagranges *Théorie des Fonctions Analytiques* is meant to form the 9th Cahier.

The contents of the first fifteen Cahiers have been already given in the Repository, vol. III. page 99, part I.

CAHIER XVI. Contains 1. The mathematical theory of capillary attraction, by Petit.—2. On the theory of the conjugate axes and of the moments of inertia of bodies, by Binet.—3 and 4. Two memoirs on Polyhedrons and Polygons, by Cauchy.—5. Researches on numbers, by the same.—6. On the graphic construction of circles determined by four conditions, by Gaultier.—7. On Definite Integrals, by Poisson.—8. On a particular case of the motion of rotation of heavy bodies, by the same.—9. On the Heliostate, by Hachette.—10. On a system of Analytic Formulae, and their application to geometry by Binet.

Cahier XVII. Contains 1. On the number of values which a function may have when the quantities it contains are changed in all possible ways, by Cauchy.—2. On functions which can have but two equal values with contrary signs, &c. by the same.—3. On the Analytic determination of a sphere which touches four others, by Binet.—4. To determine the centre and the radius of a sphere which touches four given spheres, by Hachette.—5. Experiments upon the flexibility, the force and the elasticity of Wood, by Dupin.—6. On the Resolution of Equations, by Corancez.—7. Experiments to determine the Density of the Earth, by H. Cavendish, translated from the English in the Philosophical Transactions of the Royal Society of London.—8. On the Composition of Forces and on the Decomposition of Moments, by Binet.—9. On the Oscillation of Elastic Springs, by Plana.—10. On the Aberration of the Stars, by Puissant.—11. On Material Curves of double curvature, by Binet.—12. On the number of real roots of Algebraic Equations, by Cauchy.—13. On the Integrals of Equations of Partial Differentials, by Ampere.—14. A sequel to a memoir on Definite Integrals, by Poisson.

V. MATHEMATICAL WORKS IN THE PRESS.

A supplement to the translation of LaCroix's Elementary treatise on the differential and integral calculus, by the translators of that work, being a series of examples and exercises for the use of the student, partly original and partly collected from the various transactions of learned societies and other esteemed sources.

A system of Mechanical Philosophy by the late John Robinson, L. L. D. with notes and illustrations, by Dr. Brewster, in 4 vols. 8vo.

VI. NEW BOOKS.

An Elementary Treatise on the Differential and Integral Calculus, by S. F. La Croix: translated from the French. To which is added an appendix on Finite Differences, by Mr. Herschel, 8vo.

Note. It appears from the preface that this translation is the joint labour of Messrs Babbage, Peacock and Herschel.

Algebra of the Hindus, with Arithmetic and Mensuration, Translated from the Sanscrit, by H. T. Colebrooke, Esq. 4to.

The Principles and application of Imaginary quantities. Book 1. To which are added some observations on Porisms: being the first of a series of original Tracts in various parts of the Mathematics, by Benjamin Gompertz, Esq. 4to.

An Essay on the strength and stress of Timber. Also an appendix on the strength of Iron and other Materials. By Peter Barlow of the Royal Military Academy, Woolwich.

Outlines of Natural Philosophy; being heads of Lectures delivered in the University of Edinburgh. By John Playfair, F. R. S. L. and E. Professor of Natural Philosophy in the University of Edinburgh. 2nd edit. 2 vols. 8vo.

Elements of Plane and Spherical Trigonometry, by Olinthus Gregory, L. L. D. 12mo.

Elements of Plane Geometry and Trigonometry, by John Leslie, Professor of Mathematics in the University of Edinburgh, 3rd edit. 8vo.

Philosophy of Arithmetic; by John Leslie, 8vo.

A Treatise on Spherics, by D. Cresswell, M. A. Fellow of Trinity College, Cambridge, 8vo.

An Introduction to the method of Increments, by Peter Nicholson.

The Mathematical Questions proposed in the Gentleman's Diary, and their original Answers from its commencement in the year 1741 to 1800 inclusive, 3 vols. 12mo.

The Mathematical Questions proposed in the Ladies' Diary, and their original Answers, together with some new Solutions, from its commencement in the year 1704 to 1816, by Thomas Leybourn, of the Royal Military College. 4 vols. 8vo.

Davis and Dixon's Catalogue of Scientific Books, part 1, containing the folio and quarto sizes.

VIII. Foreign Books.

ΚΛΑΥΔΙΟΥ ΠΤΟΛΕΜΑΙΟΥ ΜΑΘΗΜΑΤΙΚΕ ΣΥΝΤΑΣΙΣ (Composition Mathématique de Claude Ptolemèe), traduite, pour la première fois, du grec en francais, sur les manuscrits originaux de la Bibliothèque impériale de Paris; par M. Halma: et suivie des notes de M. Delambre, tome premier, 1 vol. in 4to. 550 pages.

Oeuvres d'Euclide, en grec, latin et français, d'après un manuscrit tres-ancien, qui était resté inconnu jusq' à nos jours; par F. Peyrard, Tome 1 et 11, 4to.

Théorie analytique des probabilités; par M. le comte La Place, 2nd edit. 4to. 600 pages.

Note. A supplement has been since published.

Essai Philosophique sur les probabiltés; par M. La Place, 8vo. 200 pages. 4to. 100 pages.

Exercices de calcul integral; sur divers ordres de Transcendantes et sur les Quadratures, 3 vols. 4to; par A. M. Legendre.

Supplément à l'Essai sur la théorie des nombres; 2nd. edition; par Legendre.

Histoire de l'Astronomie Ancienne, par Delambre, 2 vol. 4to.

Traité complet d'astronomie théorique et pratique; par M. Delambre; Trois gros volumes. 4to.

Abrégé du même Ouvrage; 8vo. 700 pages.

Traité du Calcul différentiel et Intégral, 2nd edit. tome I et II, in 4to. par S. F. La Croix.

Traité élémentaire du calcul des probabilités; par S. F. La Croix, 8vo. 300 pages.

Traitè de Physique expérimentale et mathématique; par J. B. Biot, 4 vol. 8vo. 2400 pages.

Dèveloppemens de Géometrie, avec des applications à la stabilité des vaisseaux, aux déblais et remblais, aux défilemens, &c.; par Ch. Dupin, 4to. 400 pages.

Physique Mécanique; par E. G. Fischer, traduite de l'allemand, avec des notes de M. Biot, 8vo. 500 pages.

Principes de Mathématiques de feu Joseph Anastase du Cunha, professeur à l'université de Coimbre, traduits littéralement du portugais; par J. M. d'Abreu; nouvelle édition, 8vo. 300 pages.

Elémens de Mécanique, par J. L. Boucharlat, 8vo. 350 pages.

Elémens de calcul différentiel et de calcul intégral; par J. L. Boucharlet, 8vo. 250 pages.

Réflexions sur la métaphysique du calcul infinitésimal, 2nd edit. in 8vo. par Carnot.

Nouvelles Tables d'aberration et de nutation, pour quatorze cent quatre étoilles; avec une table générale d'aberration pour les Planètes et les Comètes, &c. par le Baron de Zach.

L'attraction des montagnes et ses effets sur les fils à plomb ou sur les niveaux des instrumens d'astronomie; par le Baron de Zach, 2 vol. 8vo.

Mémoire de M. le Baron de Zach, sur le degré du méridien mesuré en Piémont, par le P. Beccaria, 4to.

Mémoire sur diverses intégrales définies; par M. G. Bidone; 4to. 120 pages.

Mémoire sur les intégrales définies; par M. Plana; 4to. 45 pages. Cet intéressant mémoire forme un utile complément aux travaux de M. M. Lagrange, Legendre, Poisson et Bidone sur le même sujet.

Mémoire sur divers problèmes de probabilité, lu à l'académie de Turin; par M. Plana.

Mémoire sur le mouvement de rotation d'un corps solide libre, autour son centre de masse, par J. J. Français.

Mémoire sur le cercle qui en touche trois autres sur un plan, et sur la sphere qui en touche quatre autres dans l'espace; par J. D. Gergonne, 4to. pp. 20. Turin.

Théorie de la distance d'un point à un autre, sur la surface d'un solide de révolution; par M. B. Goudin, 4to.

Traités élémentaires de calcul différentiel et de calcul intégral, indépendans de toutes notions de quantités infinitésimales et de limites; par M. J. B. E. Du Bourguet, 2 vol. 8vo.

Elémens de Statique, par J. B. Labey, 8vo.

Table des diviseurs pour tous les nombres du troisième million, par J. C. Burckhardt, 4to.

ARTICLE III.

Solutions to Questions proposed in Number XIV.

I. QUESTION 371, *by the Rev. Mr.* W. WOOD.

Given to find x and y, the two equations
$$3x^2 - y^2 = a, \text{ and } x^3 - y^3 - 2xy^2 = b.$$

FIRST SOLUTION, *by the Rev. Mr.* W. WOOD, *the Proposer.*

Put $y = 3z$, $x = w - 2z$, then our two equations become
$$w^2 + z^2 - 4wz = \tfrac{1}{3}a, \qquad w^3 + z^3 - 6wz(w + z) = b:$$
Make now $w + z = m$, and $wz = n$, then our two last equations become
$$m^2 - 6n = \tfrac{1}{3}a, \qquad m^3 - 9mn = b,$$
from the first $n = \dfrac{m^2 - \frac{1}{3}a}{6}$, which substituted in the other, it becomes
$$m^3 - am + 2b = 0,$$
from which m is known, then $n = \dfrac{m^2 - \frac{1}{3}a}{6}$ is also known, and w and z will be the two roots of the quadratic equation $w^2 - mw + n = 0$; lastly $y = 3z$, and $x = w - 2z$.

SECOND SOLUTION, *by* A.

Given $3x^2 - y^2 = a$, and $x^3 - y^3 - 2xy^2 = b$.

Put $x + y = 2u$, and $x - y = 2z$; then $x = u + z$, and $y = u - z$; these values being substituted for x and y in the given equations, they become
$$2u^2 + 8uz + 2z^2 = a,$$
and
$$-2u^3 + 8u^2z + 2uz^2 = b.$$
Let the first be multiplied by u, and the second subtracted from the product; the result will be
$$4u^3 = au - b,$$
or
$$4u^3 - au + b = 0.$$
Find the value of u from this cubic equation, and then the value of z may be found from the quadratic $2u^2 + 8uz + 2z^2 = a$. Consequently the values of x and y will then become known.

II. QUESTION 372, *by* B. A.

Let C be the circumference of a circle, D the diameter, c the chord of any arch a; then

$$\frac{4a\text{D}\,(\text{C} - a)}{\frac{5}{4}\text{C}^2 - (\text{C} - a)a} = c, \text{ nearly.}$$

Required the investigation.

SOLUTION, *by* A. B, *the Proposer*.

This Theorem is given in the Preface to the Bija Ganita, or Hindoo Algebra, translated by E. Strachey, Esq. It seems to be derived from the Hindoo method of computing the sines, which we find in a paper by S. Davis, Esq. in the 2nd vol. of the Asiatic Researches. Mr. Davis has detailed the process which is thus: The quadrant is divided into 24 equal parts of 225′ each, and 225 is assumed for the length of its sine (the sine of 3° 45′):—then 225 divided by 225 gives 1 (the 2d difference) which taken from 225 leaves 224 (the 1st difference); and $224 + 225 = 449$ the sine of twice the arc 3° 45′. Next 449 divided by 225 gives 2 the integral quotient (the 2d diff.) which subtracted from 224 leaves 222 (the 1st diff.); and $222 + 449 = 671$ the sine of 3 times 3° 45′. Again, 671 divided by 225 gives the integral quotient 3, which taken from 222 leaves 219; and $219 + 671 = 890$ the sine of 4 times 3° 45′. In this manner by adding the 1st and 2d integral differences, the 24 sines of the quadrant are completed.*

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* Professor *Leslie* in the notes to his Geometrical Analysis gives the following investigation: "The successive differences of the sines of the arcs $\text{A} - \text{B}$, A, and $\text{A} + \text{B}$, are $\sin \text{A} - \sin(\text{A} - \text{B})$, and $\sin(\text{A} + \text{B}) - \sin \text{A}$; and consequently the differences between these again, or the second difference of the sines, is $\sin(\text{A} + \text{B}) + \sin(\text{A} - \text{B}) - 2 \sin \text{A} = -2 \operatorname{vers} \text{B} \sin \text{A}$. The second differences of the progressive sines are hence subtractive, and always proportional to the sines themselves. Wherefore the sines may be deduced from their second differences, by reversing the usual process, and recompounding their separate elements. Thus the sines of $\text{A} - \text{B}$, and A being already known, their second and descending difference, as it is thus derived from the sine of A, will combine to form the succeeding sine of $\text{A} + \text{B}$, which is $-2 \operatorname{vers} \text{B} \sin \text{A} + (\sin \text{A} - \sin(\text{A} - \text{B})) + \sin \text{A}$."——Now the diameter being 6876, and taking 225 for the chord of 225′, we have $\frac{2 \times 225^2}{6876} = 14{\cdot}7$ nearly $=$ twice the versed sine, this is $= \frac{1}{233}$ of the radius, and is the constant multiplier of the sine; but as 225 differs but little from 233, the fraction $\frac{1}{225}$ is assumed for the multiplier, or 225 for the divisor.

Mr. *Leslie* calls this "an elegant and very ingenious mode of forming the approximate sines:" but his antipathy to the Bramins appears so inveterate, that he will not on any account allow them to be the inventors; yet he is not able to discover any author from whom they could have obtained the information.

Arcs.	Sines.	1st diff.	2d diff.
3° 45′	225	224	1
7 30	449	222	2
11 15	671	219	3
15 0	890	215	4
18 45	1105	210	5
&c.		&c.	

Now it appears from the process that each column of differences is a *recurring series:* and that any sine is equal to the sum of all the preceding differences: thus $224 + 222 + 1 + 2 = 449$;——$224 + 222 + 219 + 1 + 2 + 3 = 671$, &c. And when the table is extended to 24 arcs the sum is 3438 the sine of 90° or half the diameter.

If that part of the quadrant whose sine is required be denoted by n (the quadrant being divided into 24 equal parts) then the whole sum of the 24 terms of the two series of differences being $= \frac{1}{2}\text{D}$ ($\text{D} =$ the diam.) the sum of n terms of the series will be $\frac{(48 - n)\,4n}{48^2 + (24 - n)^2} \times \frac{1}{2}\text{D}$ nearly, the sine of the arc which is n parts of the quadrant; and its double or $\frac{(48 - n)\,4n}{48^2 + (24 - n)^2} \times \text{D}$ is the chord of twice that arc.

But if the circumference be denoted by 96 ($= \text{C}$) then n will represent the arc itself, and the expression for the chord becomes $\frac{(\frac{1}{2}\text{C} - n)\,4n}{\frac{1}{4}\text{C}^2 + (\frac{1}{4}\text{C} - n)^2} \times \text{D} = \frac{(\frac{1}{2}\text{C} - n)n}{\frac{1}{4}\text{C}^2 + (\frac{1}{4}\text{C} - n)^2} \times 4\text{D} = \frac{\frac{1}{2}\text{C}n - n^2}{\frac{1}{4}\text{C}^2 + \frac{1}{16}\text{C}^2 - \frac{1}{2}\text{C}n + n^2} \times 4\text{D}$; and multiplying both numerator and denominator by 4 gives $\frac{(\text{C} - 2n)\,2n}{\frac{5}{4}\text{C}^2 - (\text{C} - 2n)\,2n} \times 4\text{D}$, which is the proposed theorem when a is put for the arc $2n$.

It may, however, be remarked that the theorem brings out the chords too great when the arcs are small; and too little when they are about $\frac{1}{4}$ of the circumference.

III. QUESTION 373, *by* Z.

Find what conditions must have place among the coefficients of $x^3 - px^2 + qx - r = 0$, that the roots may be in harmonical progression, and find those roots.

FIRST SOLUTION, *by Mr.* JOHN WALLACE, *R. M. College.*

Let a, b, c be the roots of the proposed equation. Then, by the theory of equations, we have

$$a + b + c = p,\ ab + ac + bc = q.\ abc = r.$$

But since by the conditions of the question a, b, c are in harmonical progression, so that $a : c :: a - b : b - c$; it is evident that $ab + bc = 2ac$, and therefore we have

$(ab + bc + ac) \times b = 3abc$, that is $bq = 3r$; wherefore $b = \frac{3r}{q}$.

Again, from the equation $ab + bc = 2ac$, we obtain $3ac = ab + bc + ac = q$; or $ac = \frac{1}{3}q$. But the equation $ab + bc = 2ac$ gives also

$$a + b + c = \frac{2ac}{b} + b = \frac{2ac + b^2}{b}.$$

Substituting therefore for $a + b + c$, for b and for ac their respective values, we obtain

$$p = \frac{2q^3 + 27r^2}{9qr},$$

and this equation expresses the relation which must subsist among the coefficients of the given equation, in order that the roots may be in harmonical progression.

One of the roots of the equation has already been found, namely $b = \frac{3r}{q}$; and from the equations $a + c = p - b = p - \frac{3r}{q}$ and $ac = \frac{1}{3}q$ we obtain the other two roots,

$$a = \tfrac{1}{2}\left(p - \frac{3r}{q}\right) + \tfrac{1}{2}\sqrt{\left\{\left(p - \frac{3r}{q}\right)^2 - \frac{4q}{3}\right\}}$$

$$c = \tfrac{1}{2}\left(p - \frac{3r}{q}\right) - \tfrac{1}{2}\sqrt{\left\{\left(p - \frac{3r}{q}\right)^2 - \frac{4q}{3}\right\}}.$$

SECOND SOLUTION, *by* A.

Let u, v and w be the roots required.

By the nature of harmonical progression,

$$u : w :: u - v : v - w,$$

$$\text{or } uv - uw = uw - vw \ldots\ldots\ldots\ldots\ldots \quad (1)$$

and by the nature of equations

$$u + v + w = p \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$uv + uw + vw = q \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$uvw = r \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

From these equations we have to determine the values of u, v and w; and the relation between the coefficients p, q and r.

From equation (1), $(u+w)v = 2uw$,
and from equation (3), $(u+w)v = q - uw$.

Therefore $2uw = q - uw$, or $uw = \frac{q}{3}$;

and by equation (4) $uvw = r = \frac{qv}{3}$; therefore $v = \frac{3r}{q}$, and

consequently $u + w = \frac{2uw}{v} = \frac{2q^2}{9r}$.

We have now the two equations $uw = \frac{q}{3}$, and $u + w = \frac{2q^2}{9r}$,

and from these we immediately find

$$u = \frac{q^2}{9r} \pm \frac{1}{9r}\sqrt{(q^4 - 27qr^2)}$$

$$w = \frac{q^2}{9r} - \frac{1}{9r}\sqrt{(q^4 - 27qr^2)}.$$

These values of u, v, and w being substituted in equation (2) we get

$\frac{2q^2}{9r} + \frac{3r}{q} = p$, for the relation of the coefficients.

IV. QUESTION 374, *by* JULIUS.

Exponential equations of the form $x^x = a$, may be divided into three classes, viz. those having only one real root, those having two real roots, and those which have no real root: It is required to point out the limits; and, in the case of two real roots, to shew what functions they are of each other?

FIRST SOLUTION, *by* JULIUS, *the Proposer.*

It is obvious in the first place, that if $a > 1$, the equation can have but one real root, for let $x^x = a$, a being > 1, then x must also be > 1; and

put this in the form ......$x \times \log x = \log a$,
and if possible let also ....$x' \times \log x' = \log a$,
where also x' must be greater than 1, but this is impossible, for if $x' > x$, then $\log x' > \log x$, and consequently the product of the two latter cannot be equal to the product of the two former, and the same may be shewn to be true, if we suppose

$x' < x$. Since then x' can be neither greater nor less than x, it must be equal to it, that is, the equation can have but one root when a is > 1.

Again, if we find that value of x which gives x^x a minimum, we shall have $x =$ nat. num. to hyp. log. of $-1 = \cdot 36788$ which gives $a = \cdot 6922$, therefore when a is less than $\cdot 6922$ the equation has no real root; but if a be $> \cdot 6922$ and < 1; it will have two real roots, the relation of which to each other may be found as follows:

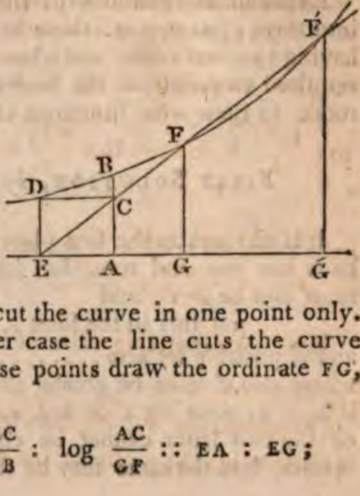
Let x and rx be the two roots, so that $(xr)^{xr} = x^x$; then $x^r \times r^r = x$, or $x^{r-1} \times r^r = 1$, or $x^{r-1} = \frac{1}{r^r}$, or $x = \frac{1}{r^{\frac{r}{r-1}}}$,

or $x = r^{\frac{r}{1-r}}$. Whence x being first found, r may be determined, also since x and rx are the two roots, and since $x = r^{\frac{r}{1-r}}$, we have $rx = r^{\frac{1}{1-r}}$.

So that the roots are to each other as $r^{\frac{r}{r-1}}$ to $r^{\frac{1}{r-1}}$.

SECOND SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let DBF be a logarithmic curve, of which EA is the axis, and AB the ordinate that represents unity. In either of the angles which this line makes with the axis describe a rectangle ACDE. Draw the diagonal EG through the other two angles: and if the points C and A are on the same side of the curve, this line may either pass entirely on one side of the curve, or may touch it, or may cut it in two points, because the curve is concave all one way: but if C and A are on opposite sides, the line will cut the curve in one point only. Let us suppose that in the former case the line cuts the curve in F and F′; from either of these points draw the ordinate FG, and by the nature of the curve,



$$\log \frac{\text{ED}}{\text{AB}} : \log \frac{\text{ED}}{\text{GF}} \text{ or } \log \frac{\text{AC}}{\text{AB}} : \log \frac{\text{AC}}{\text{GF}} :: \text{EA} : \text{EG};$$

But by similar triangles

$$\mathrm{EA} : \mathrm{EG} :: \mathrm{AC} : \mathrm{GF} :: \frac{\mathrm{AC}}{\mathrm{GF}} : 1,$$

Therefore, $\log \frac{\mathrm{AC}}{\mathrm{AB}} : \log \frac{\mathrm{AC}}{\mathrm{GF}} :: \frac{\mathrm{AC}}{\mathrm{GF}} : 1$,

and $\frac{\mathrm{AC}}{\mathrm{GF}} \times \log \frac{\mathrm{AC}}{\mathrm{GF}} = \log \frac{\mathrm{AC}}{\mathrm{AB}}$;

Now AB being $= 1$, let AC $= a$, and $\frac{a}{\mathrm{GF}} = x$, the above equation then becomes

$$x \times \log x = \log a$$

$$\text{and therefore, } x^x = a.$$

Hence we have this construction to find the value or values of x in the equation $x^x = a$.

1. Make AC $= a$, and describe the rectangle ACDE, so that D the angle opposite to A may be on the curve.

2. Draw the diagonal EC, and produce it, if necessary, to meet the curve in two points F, F′, or one point F. Draw the ordinates FG, F′G′ (supposing that there are two intersections) and the values of x will be $x = \frac{a}{\mathrm{FG}}$, $x' = \frac{a}{\mathrm{F'G'}}$.

If AC $= a$ is of such a magnitude that the line EC just touches the curve there will be only one solution, and if a be less, the problem will be impossible. When ECF is a tangent, then EG will be the subtangent; this, in every logarithmic curve, is the *modulus* of the system of logarithms; or it is the logarithm of $e = 2{\cdot}7\ldots\ldots$ the radical number in Napier's system. Since, in this case, EG $= \log e$, and since also, EG $= \log \frac{\mathrm{GF}}{\mathrm{AC}} = \log \frac{\mathrm{GF}}{a}$, it follows that $\frac{\mathrm{GF}}{a} = e$, and $x = \frac{a}{\mathrm{GF}} = \frac{1}{e}$ and the minimum value of $a = x^x = \left(\frac{1}{e}\right)^{\frac{1}{e}} = \frac{1}{e^{\frac{1}{e}}}$.

The corresponding value of the ordinate GF is $\frac{a}{x} = e^{1-\frac{1}{e}}$.

Suppose now a to begin with the value $e^{-\frac{1}{e}}$, and afterwards to increase, the ordinate GF will at first have one value, viz.

$e^{1-\frac{1}{e}}$, and afterwards will have two, one increasing and another decreasing; corresponding to these, x will begin with the single value $\frac{1}{e}$, and then will have two, one decreasing and another increasing: when a has increased to 1, one value of the ordinate will be $=$ AB $= 1$ and the other infinite, the corresponding values of x will be $\frac{1}{1} = 1$ and $\frac{1}{\infty} = 0$.

Let us now suppose that a increases beyond AB $= 1$, then there will only be one intersection, and one value of the ordinate, therefore for every value of a greater than 1, x will have a single value which will go on increasing continually, to infinity, as x increases, as is sufficiently evident from the nature of the function x^x.

V. QUESTION 375, *by Mr.* CUNLIFFE, *R. M. College.*

It is required to find two such rational fractions, that the cube of either being added to the square of the other, shall make the same sum; and furthermore, that their sum and sum of their squares may both be square numbers.

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let the two fractions be denoted by $\frac{a^2x}{x+y}$ and $\frac{a^2y}{x+y}$; for the sum of these is, $\frac{a^2(x+y)}{x+y} = a^2 =$ a square. And the sum of their squares is $\frac{a^4(x^2+y^2)}{(x+y)^2}$, which will evidently be a square, when x^2+y^2 is a square.

Again, by the question, the cube of either of the fractions, being added to the square of the other must make the same sum, that is $\frac{a^6x^3}{(x+y)^3} + \frac{a^4y^2}{(x+y)^2} = \frac{a^6y^3}{(x+y)^3} + \frac{a^4x^2}{(x+y)^2}$, which easily reduces to $a^2(x^3-y^3) = (x+y)(x^2-y^2)$, whence $a^2 = \frac{(x+y)(x^2-y^2)}{x^3-y^3} = \frac{(x+y)^2}{x^2+xy+y^2}$, and therefore $x^2 + xy + y^2$ must be a square.

The question has, therefore, been reduced to the finding of such values of x and y as will make $x^2 + y^2$ and $x^2 + xy + y^2$ both rational squares.

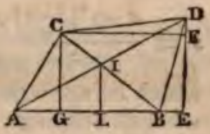
In the solution of Question 332, No. XIV. of the Repository, it has been found that 2415 and 1768 are such values of x and y as will make the expressions $x^2 + y^2$ and $x^2 + xy + y^2$, both rational squares; whence and from what has been deduced $a^2 = \frac{(x+y)^2}{x^2+xy+y^2} = \frac{(4183)^2}{(3637)^2}$; and hence $\frac{a^2x}{x+y} = \frac{2415 \times 4183}{(3637)^2} = \frac{10101945}{13227769}$, and $\frac{a^2y}{x+y} = \frac{1768 \times 4183}{(3637)^2} = \frac{7395544}{13227769}$, are two fractions that will answer the question.

VI. QUESTION 376, *by Mr.* CUNLIFFE.

Find the equation of the curve which is the locus of the intersection of the diagonals of a trapezium whose sides are given in length, and one of the sides given by position.

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let ACDB be a trapezium, all the sides of which are given in length, and let the side AB be also given by position. Draw the diagonals AD, BC intersecting each other in I, and draw CG, IL, and DE at right angles to AB; also draw CF parallel to AB, meeting DE in F.



Put AB $= a$, AC $= b$, CD $= c$, DB $= d$, AL $= x$, IL $= y$, AG $= z$, and BE $= v$. Then CG $= \sqrt{(b^2 - z^2)}$, DE $= \sqrt{(d^2 - v^2)}$, CF $=$ GE $= a + v - z$, and DF $=$ DE $-$ EF $= \sqrt{(d^2 - v^2)} - \sqrt{(b^2 - z^2)}$: and CF^2 + DF^2 = CD^2, that is

$$(a+v-z)^2 + \left\{\sqrt{(d^2-v^2)} - \sqrt{(b^2-z^2)}\right\}^2 = c^2 \ldots\ldots (a).$$

Because of the parallels LI, DE:

$$\text{AL : LI :: AE : ED} = \frac{\text{LI} \times \text{AE}}{\text{AL}} = \frac{y(a+v)}{x} = \sqrt{(d^2 - v^2)};$$

Also, because of the parallels LI, CG;

$$\text{BL : LI :: BG : CG} = \frac{\text{LI} \times \text{BG}}{\text{BL}} = \frac{y(a-z)}{a-x} = \sqrt{(b^2 - z^2)}.$$

And hence $\sqrt{(d^2 - v^2)} - \sqrt{(b^2 - z^2)} = \frac{y(a+v)}{x} - \frac{y(a-z)}{a-x}$;
by means of which equation (a) becomes

$$(a + v - z)^2 + y^2 \left\{ \frac{a+v}{x} - \frac{a-z}{a-x} \right\}^2 = c^2 \ldots\ldots\ldots(b).$$

Again, from the equation $\frac{y(a+v)}{x} = \sqrt{(d^2 - v^2)}$, we get

$$v = \frac{x\sqrt{\left\{ d^2x^2 - y^2(a^2 - d^2) \right\}} - ay^2}{x^2 + y^2},$$

and from the equation $\frac{y(a-z)}{a-x} = \sqrt{(b^2 - z^2)}$, we get

$$z = \frac{ay^2 - (a-x)\sqrt{\left\{ b^2(a-x)^2 - y^2(a^2 - b^2) \right\}}}{(a-x)^2 + y^2};$$

and these being written for v and z in equation (b) there will result an equation in terms of x and y and given quantities, which will be the required equation of the locus of I. The general equation of the curve will be complex, as is obvious from what has been done.

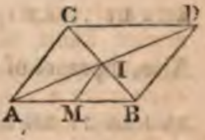
When $a = b = d$, then

$$v = \frac{x\sqrt{\left\{ d^2x^2 - y^2(a^2 - d^2) \right\}} - ay^2}{x^2 + y^2} = \frac{a(x^2 - y^2)}{x^2 + y^2}, \text{ and}$$

$$z = \frac{a - y^2(a-x)\sqrt{\left\{ b^2(a-x)^2 - y^2(a^2 - b^2) \right\}}}{(a-x)^2 + y^2} = \frac{a\left\{ y^2 - (a-x)^2 \right\}}{(a-x)^2 + y^2},$$

which being written respectively for v and z in equation (b), we shall obtain an equation of the curve. But even in this case, the dimensions of x and y, in the resulting equation, are high and complex.

In the particular case where the opposite sides of the trapezium are equal; that is, when the trapezium is a parallelogram, the locus of the intersection of the diagonals will be a circle, having its centre in the middle of the side which is given by position; and its diameter equal to one of the other two parallel sides.

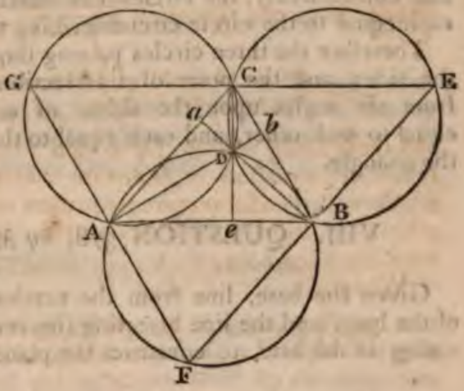


For let ACDB be a parallelogram: the diagonals AD, BC bisect each other in I, as is very well known. Draw IM parallel to AC; then M is the middle of AB, and MI $= \frac{1}{2}$AC; wherefore the locus of I, will be a circle whose centre is M, and radius MI $= \frac{1}{2}$AC.

VII. QUESTION 377, *by Mr.* CUNLIFFE.

What is the relation of the diameters of the three circles, passing through the extremities of the sides, and point of intersection of the perpendiculars from the angles upon the sides of a plane triangle?

FIRST SOLUTION, *by a* LADY.

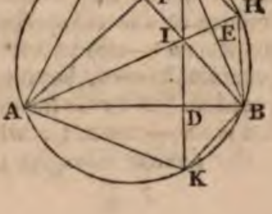


Let ABC be the triangle, D the point of intersection of the perpendiculars from the angles upon the sides, and CEB, CGA, AFB the circles passing through D and the extremities of the sides.

Draw the diameters DE, DF, DG, and join EB, BF, FA, AG, GC, CE.

Since DBE is in a semicircle it is a right angle; for the same reason DBF is a right angle; therefore EBF is a straight line: and in like manner it may be shown that GE, GF are straight lines. Now it is evident that the triangles BA*b*, BC*e* are similar, for the angles *b* and *e* are right angles and B is common to both, therefore the angle BC*e* is equal to BA*b*. But BC*e* or BCD is equal to BED being in the same segment, for that reason also DAB = DFB; hence DFB = DEB and DE = DF. In the same way it easily appears that DF = DG, hence we have DE = DF = DG; and thus the ratio of the diameters is that of equality.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*



Let ACB be a plane triangle circumscribed by a circle; AE, BF, and CD perpendiculars from the angles upon the opposite sides: These perpendiculars will intersect each other in the same point I, as is very well known.

Produce CD to meet the circumference of the circle again in K, and join AK, BK. The angle BCD (BCK)

$=$ BAE (DAI) being both complements of ABE. Also the angle BAK (DAK) $=$ BCK, as both angles stand upon the same arc BK: therefore the angle DAI $=$ the angle DAK, whence DI $=$ DK. Wherefore the triangle AIB is equal and similar to the triangle AKB; and therefore the circle circumscribing the triangle AIB will be equal to the circle circumscribing the triangle AKB, that is, equal to the circle circumscribing the triangle ACB.

And by producing the perpendiculars BF, AE, till they meet the circle again in the points G and H, and drawing AG, CG; BH, HC, we may, in the same manner, exactly, prove, that the triangles AIC, BIC are respectively equal to the triangles AGC, and BHC; and consequently, the circles circumscribing these triangles are each equal to the circle circumscribing the triangle ACB.

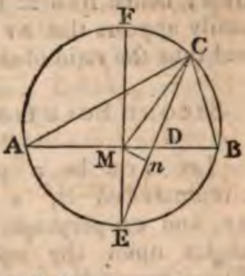
Therefore the three circles passing through the extremities of the sides, and the point of intersection of the perpendiculars from the angles upon the sides, of any plane triangle, are equal to each other, and each equal to the circle circumscribing the triangle.

VIII. QUESTION 378, *by Mr.* CUNLIFFE.

Given the base, line from the vertical angle to the middle of the base, and the line bisecting the vertical angle and terminating in the base, to construct the plane triangle.

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let ACB represent the required triangle circumscribed by a circle, EF being a diameter bisecting the base AB in M: draw EC, cutting AB in D; also draw Mn perpendicular to EC. By the known property of the circle $DC \times DE = DC \times (Dn + En) = AD \times DB = (AM + MD) \times (AM - MD) = AM^2 - MD^2$; whence $DC \times (Dn + En) + MD^2 = AM^2$. Again, by a well known property of triangles, $DC^2 + MD^2 + 2DC \times Dn = MC^2$; whence $2DC \times Dn + MD^2 = MC^2 - DC^2$; and the difference of the two preceding results is $DC \times (En - Dn) = AM^2 + CD^2 - MC^2$, and hence $En - Dn$ becomes known, because AM, MC and CD are all given by the question. Now let the difference of En and Dn be denoted by L; that is $En - Dn = L$; whence $En + Dn = DE = L + 2Dn$.



Per similar triangles $MD : Dn :: DE : MD$, whence $MD^2 =$

D*n* × DE, and this being written for MD^2, in the property DC × DE + MD^2 = AM^2, before deduced, that property becomes DC × DE + D*n* × DE = (DC + D*n*) × DE = AM^2, and here again for DE writing its equal L + 2D*n*, the last expression becomes (DC + D*n*) × (L + 2D*n*) = AM^2, which is an obvious and easy case of the determinate section, from whence D*n* may be found; and consequently E*n* = L + 2D*n*, is given or known: Now D*n* and E*n* being known, the right-angled triangle EMD is entirely known. Wherefore having formed this triangle, produce ED till DC is equal to the given line bisecting the vertical angle; join AC BC, and ACB will be the required triangle, as is evident from the analysis.

The following property, deduced in the foregoing solution, deserves to be noticed: viz.

ACB is a triangle circumscribed by a circle, EF being a diameter bisecting the base AB in M; draw the chord EC, cutting AB in D, and draw M*n* perpendicular to EC; then C*n* × DE = AM^2. For we have found (DC + D*n*) × DE = C*n* × DE = AM^2.

IX. QUESTION 379, *by* PALABA.

Given that the distance of the centre of gravity of an area from its vertex is an *n*th part of the abscissa, to find the distance of the centre of gravity of the solid generated by the same area revolving round its axis.

SOLUTION, *by* PALABA, *the Proposer.*

By hypothesis $\frac{\int yx\dot{x}}{\int y\dot{x}} = \frac{x}{n}$ $\therefore \int yx\dot{x} = \frac{x}{n}\int y\dot{x}$

$\therefore yx\dot{x} = \frac{\dot{x}}{n}\int y\dot{x} + \frac{yx\dot{x}}{n}$ $\therefore nyx\dot{x} = \dot{x}\int y\dot{x} + yx\dot{x}$

$\therefore (n-1)\,xy = \int y\dot{x}$ $\therefore (n-1)\,x\dot{y} + (n-1)\,y\dot{x} = y\dot{x}$

$\therefore (n-1)\,x\dot{y} = (2-n)y\dot{x}$ $\therefore x\dot{y} = \frac{2-n}{n-1}\cdot y\dot{x}$

$$\therefore \int xy\dot{y} = \frac{2-n}{n-1}\int y^2\dot{x}.$$

But $\int xy\dot{y} = \frac{xy^2}{2} - \int\frac{y^2\dot{x}}{2}$ $\therefore \frac{2-n}{n-1}\int y^2\dot{x} = \frac{xy^2}{2} - \int\frac{y^2\dot{x}}{2}$

$$\therefore \left(\frac{2-n}{n-1}+\tfrac{1}{2}\right)\int y^2\dot{x} = \frac{y^2x}{2}, \text{ or } \frac{3-n}{n-1}\int y^2\dot{x} = y^2x$$

$$\therefore \int y^2\dot{x} = \frac{n-1}{3-n} \cdot y^2x.$$

Again $x\dot{y} = \frac{2-n}{n-1} \cdot y\dot{x} \therefore \int x^2y\dot{y} = \frac{2-n}{n-1}\int y^2x\dot{x}.$

But $\int x^2y\dot{y} = \frac{x^2y^2}{2} - \int y^2x\dot{x} \therefore \frac{2-n}{n-1}\int y^2x\dot{x} = \frac{x^2y^2}{2} - \int y^2x\dot{x}$

$$\therefore \frac{1}{n-1}\int y^2x\dot{x} = \frac{x^2y^2}{2} \therefore \int y^2x\dot{x} = \frac{n-1}{2} \cdot x^2y^2$$

$$\therefore \frac{\int y^2x\dot{x}}{\int y^2\dot{x}} = \frac{3-n}{2} \cdot x.$$

X. QUESTION 380, *by* PALABA.

Find the distance from the vertex of the centre of gravity of the area of the catenaria without the aid of logarithms.

SOLUTION, *by* PALABA, *the Proposer.*

The distance required $= \frac{\int yx\dot{x}}{\int y\dot{x}}.$

In this curve $z\dot{y} = a\dot{x}$, also $z^2 = 2ax + x^2$, $\therefore z\dot{z} = a\dot{x} + x\dot{x}$

$$\therefore \dot{z} = \frac{a\dot{x}}{z} + \frac{x\dot{x}}{z} \therefore \frac{x\dot{x}}{z} = \dot{z} - \frac{a\dot{x}}{z} = \dot{z} - \dot{y}.$$

Now the $\int yx\dot{x} = \frac{yx^2}{2} - \tfrac{1}{2}\int x^2\dot{y}.$

But $\int x^2\dot{y} = \int x \cdot x\dot{y} = \int x \cdot \frac{ax\dot{x}}{z} = \int ax(\dot{z} - \dot{y}) = a\int x\dot{z} - a\int x\dot{y}$;

$$\therefore \int yx\dot{x} = \frac{x^2y}{2} + \frac{a}{2}\int x\dot{y} - \frac{a}{2}\int x\dot{z}.$$

Again $\int x\dot{z} = xz - \int z\dot{x} = xz - \int\frac{z^2\dot{y}}{a} = xz - \frac{1}{a}\int(2ax\dot{y} + x^2\dot{y}) = xz - 2\int x\dot{y} - \frac{1}{a}\int x^2\dot{y}.$

$$\therefore \int yx\dot{x} = \frac{x^2y}{2} + \frac{a}{2}\int x\dot{y} - \frac{axz}{2} + a\int x\dot{y} + \tfrac{1}{2}\int x^2\dot{y}$$

$$= \frac{x^2y}{2} + \frac{3a}{2}\int x\dot{y} - \frac{axz}{2} + \tfrac{1}{2}\int x^2\dot{y}.$$

But $\int x^2\dot{y} = x^2y - 2\int yx\dot{x}$,

$$\therefore \int yx\dot{x} = \frac{x^2y}{2} - \frac{axz}{2} + \frac{x^2y}{2} + \frac{3a}{2}\int x\dot{y} - \int yx\dot{x}$$

$$\therefore 2\int yx\dot{x} = x^2y - \frac{axz}{2} + \frac{3a}{2}\int\frac{ax\dot{x}}{z}$$

$$\therefore \int yx\dot{x} = \frac{x^2y}{2} - \frac{axz}{4} + \frac{3a^2}{4}(z-y).$$

Also $\int y\dot{x} = xy - \int x\dot{y} = xy - az + ay$

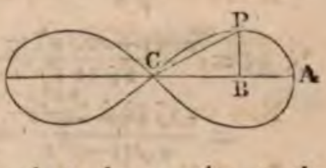
$$\therefore \frac{\int yx\dot{x}}{\int y\dot{x}} = \tfrac{1}{4}\cdot\frac{2x^2y - axz + 3a^2z - 3a^2y}{xy - az + ay} = \text{by division}$$

$$\frac{x-a}{2} + \frac{a}{4}\cdot\frac{xz + a(z-y)}{xy - a(z-y)}.$$

XI. QUESTION 381, *by* PALABA.

The equation to the lemniscata being $(x^2 + y^2)^2 = x^2 - y^2$; find its area contained between the values of $x = 0$ and $= 1$.

FIRST SOLUTION, *by a* LADY.



Let CPA be the lemniscata, C its centre, CA its semi-axis, and PB an ordinate: then CB $= x$, PB $= y$; join CP and put r for the variable radius CP, unity for the semi-axis CA, then the equation $(x^2 + y^2)^2 = x^2 - y^2$ becomes $r^2 = \sqrt{(x^2 - y^2)}$. Put ϕ for the variable angle PCA, then because $x = r\cos\phi$, $y = r\sin\phi$, the equation to the lemniscata becomes $r = \sqrt{(\cos^2\phi - \sin^2\phi)}$. Now $dr =$ $-\dfrac{2d\phi\cos\phi\sin\phi}{\sqrt{(\cos^2\phi - \sin^2\phi)}}$ and $dx = dr\cos\phi - rd\phi\sin\phi$, and by substituting in the latter for r and dr we find

$$dx = -d\phi\sin\phi\sqrt{(\cos^2\phi - \sin^2\phi)} - \frac{2d\phi\cos^2\phi\sin\phi}{\sqrt{(\cos^2\phi - \sin^2\phi)}},$$

and in the same manner the equation $y = r\sin\phi$ becomes $y = \sin\phi\sqrt{(\cos^2\phi - \sin^2\phi)}$. Hence, if we substitute these values of y and dx in $\int ydx$, which is the general expression for areas, we have

$$\int ydx = \int d\phi\sin^4\phi - 3\int d\phi\sin^2\phi\cos^2\phi.$$

But it is evident that

$$\begin{aligned}\int d\varphi \sin^4\varphi &= \int (d\varphi \sin\varphi)\sin^3\varphi \\ &= -\cos\varphi\sin^3\varphi + 3\int d\varphi\cos^2\varphi\sin^2\varphi.\end{aligned}$$

Therefore we have

$$\int y\,dx = \mathrm{C} - \cos\varphi\sin^3\varphi.$$

Now since $x^2+y^2=r^2$ and $x^2-y^2=r^4$, it is evident that

$$\cos\varphi=\frac{x}{r}=\sqrt{\frac{1+r^2}{2}} \text{ and } \sin\varphi=\frac{y}{r}=\sqrt{\frac{1-r^2}{2}}.$$

Hence, by substitution, we find

$$\text{area}=\int y\,dx=\mathrm{C}-\tfrac{1}{4}(1-r^2)\sqrt{(1-r^4)}.$$

When $x=0$, we have also $r=0$, and therefore

$$\text{area}=\mathrm{C}-\tfrac{1}{4};$$

when $x=\mathrm{AC}=1$, then angle φ vanishes and r becomes equal to AC or 1; hence we have

$$\text{area}=\mathrm{C}-0.$$

Taking therefore the integral between the proposed limits $x=0$ and $x=1$, we find the area of half one of the ovals equal to $\frac{1}{4}$.

SECOND SOLUTION, *by* PALABA, *the Proposer.*

Assume $x^2+y^2=v^2 \therefore x^2-y^2=v^4$

$$\therefore\ x=\frac{v\sqrt{(1+v^2)}}{\sqrt{2}},\ y=\frac{v\sqrt{(1-v^4)}}{\sqrt{2}},\ \dot{x}=\frac{\dot{v}(1+2v^2)}{\sqrt{2}\cdot\sqrt{(1+v^2)}}$$

$$\therefore\ y\dot{x}=\frac{v\dot{v}(1+2v^2)\sqrt{(1-v^2)}}{2\sqrt{(1+v^2)}}=\frac{v\dot{v}(1+2v^2)(1-v^2)}{2\sqrt{(1-v^4)}}$$

$$=\frac{v^3\dot{v}+v\dot{v}-2v^5\dot{v}}{2\sqrt{(1-v^4)}}=\frac{v^3\dot{v}}{2\sqrt{(1-v^4)}}+\frac{4v^3\dot{v}-8v^7\dot{v}}{8\sqrt{(v^4-v^8)}}$$

$$\therefore\ \text{Area}=\mathrm{C}-\tfrac{1}{4}\sqrt{(1-v^4)}+\tfrac{1}{4}\sqrt{(v^4-v^8)}=\mathrm{C}-\frac{(1-v^2)\sqrt{(1-v^4)}}{4}$$

$$0=\mathrm{C}-\tfrac{1}{4}$$

$$\therefore\ \text{Area}=\tfrac{1}{4}-\frac{(1-v^2)\sqrt{(1-v^4)}}{4}=\tfrac{1}{4},\text{ when } x=1.$$

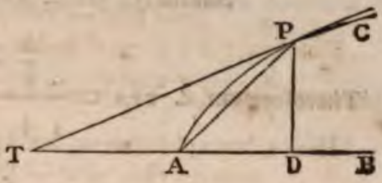
Hence it appears that the area of both ovals equals the square described on the semi-axis of the equilateral hyperbola.

XII. QUESTION 382, *by* PALABA.

Determine that point in a curve whose equation is $a^{n-1}x=y^n$ to which a line must be drawn from the vertex making the greatest angle with the curve.

FIRST SOLUTION, *by a* LADY.

Let APC be the curve, A the vertex, AB the line of x's whose origin is at A, and PD that of the y's. From P draw the tangent PT to meet the axis AB produced in T. Join AP. The angle APT must be a maximum.



Let the angle PAD $= m$ and PTD $= p$, then PAD $-$ PTD $=$ TPA $= m - p$; therefore tan TPA $= \frac{\tan m - \tan p}{1 + \tan m \tan p}$.

But $\tan m = \frac{\text{PD}}{\text{AD}} = \frac{y}{x}$ and $\tan p = \frac{dy}{dx}$, and by substituting these, we have tan TPA $= \frac{ydx - xdy}{xdx + ydy}$. Now $y^n = a^{n-1} x$ being the equation of the curve, we find $x = \frac{y^n}{a^{n-1}}$ and $dx = \frac{ny^{n-1}\,dy}{a^{n-1}}$, therefore

$$\tan \text{TPA} = \frac{(n-1)a^{n-1}\,y^{n-1}}{ny^{2n-2} + a^{2n-2}} = \text{a maximum.}$$

By taking the differential of this expression, and reducing, we obtain

$$ny^{2n-2} - a^{2n-2} = 0, \text{ or } y^{2n-2} = \frac{a^{2n-2}}{n}.$$

Consequently, $y = \frac{a}{n^{\frac{1}{2n-2}}}$, and $x = \frac{a}{n^{\frac{n}{2n-2}}}$.

Let $n = 2$, then $y^2 = ax$, the equation to the common parabola, and $y = \frac{a}{\sqrt{2}}$, $x = \frac{a}{2}$. In this case $x = \frac{1}{2}$ the parameter.

SECOND SOLUTION, *by* PALABA, *the Proposer.*

Let P be the required point, to which AD, DP are the co-ordinates. By trigonometry,

$$\text{PD } (y) : \text{DT} \left(\frac{y\dot{x}}{\dot{y}}\right) :: 1 : \tan \angle \text{ TPD} = \frac{\dot{x}}{\dot{y}};$$

$$\text{Similarly, } \tan \angle \text{ APD} = \frac{x}{y}.$$

$$\text{Therefore } \tan \angle \text{ TPA} = \frac{\frac{\dot{x}}{\dot{y}} - \frac{x}{y}}{1 + \frac{\dot{x}}{\dot{y}} \cdot \frac{x}{y}} = \text{a max.}$$

$$\text{Now } \therefore a^{n-1} x = y^{n} \therefore \frac{\dot{x}}{\dot{y}} = \frac{ny^{n-1}}{a^{n-1}};$$

$$\therefore \frac{\frac{ny^{n-1}}{a^{n-1}} - \frac{y^{n-1}}{a^{n-1}}}{1 + \frac{ny^{n-1}}{a^{n-1}} \cdot \frac{y^{n-1}}{a^{n-1}}} = \text{max.}$$

$$\text{or } \frac{(n-1)\, a^{n-1} \,.\, y^{n-1}}{a^{2n-2} + ny^{2n-2}} = \text{max.}$$

$$\therefore \frac{ny^{2n-2} + a^{2n-2}}{y^{n-1}} = \text{min.}$$

$$\therefore n(n-1)\, y^{n-2}\, \dot{y} - \frac{(n-1)\, a^{2n-2} \dot{y}}{y^{n}} = 0.$$

$$\therefore ny^{2n-2} - a^{2n-2} = 0, \therefore y = \frac{a}{n^{\frac{1}{(2n-2)}}}.$$

XIII. QUESTION 383, *by* Palaba.

Two cylinders of equal diameters and altitudes open at the top, are filled with water, and one of them is placed upon the other. A small orifice being made in the base of each, it is required to ascertain the time in which the lower cylinder will be completely emptied.

Solution, *by* Palaba, *the Proposer.*

Let r = radius of the base, h = altitude, a = area of orifice, x and z = altitudes of the surfaces of the fluid in the lower and upper cylinders, when the time, t, is elapsed; $g = 32\frac{1}{6}$ feet. Then the velocity at the orifice being that due to half the depth, $a\sqrt{(gx)}$ and $a\sqrt{(gz)}$ are the quantities respectively discharged per second through the two orifices, $\therefore a\sqrt{g} \cdot (\sqrt{x} - \sqrt{z}) \cdot t =$ quantity by which the surface in the lower cylinder descends in the time (t);

$$\therefore a\sqrt{g} \cdot (\sqrt{x} - \sqrt{z}) \cdot \dot{t} = -\pi r^2 \dot{x} \quad \therefore \dot{t} = -\frac{\pi r^2}{a\sqrt{g}} \cdot \frac{\dot{x}}{\sqrt{x} - \sqrt{z}},$$

but $\dot{t} = -\frac{\pi r^2}{a\sqrt{g}} \cdot \frac{\dot{z}}{\sqrt{z}} \therefore \frac{\dot{x}}{\sqrt{x} - \sqrt{z}} = \frac{\dot{z}}{\sqrt{z}} \therefore z^{\frac{1}{2}} \dot{x} = x^{\frac{1}{2}} \dot{z} - z^{\frac{1}{2}} \dot{z}.$

Assume $x = vz \therefore \dot{x} = v\dot{z} + z\dot{v} \therefore$ by substitution

$$z^{\frac{1}{2}} \cdot (v\dot{z} + z\dot{v}) = v^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z} - z^{\frac{1}{2}} \dot{z} \therefore v\dot{z} + z\dot{v} = v^{\frac{1}{2}} \dot{z} - \dot{z}$$

$$\therefore (v - \sqrt{v} + 1)\dot{z} + z\dot{v} = 0 \therefore \frac{\dot{z}}{z} + \frac{\dot{v}}{v - \sqrt{v} + 1} = 0; \text{ let } v - \tfrac{1}{2} = y$$

$$\therefore v - \sqrt{v} + 1 = y^2 + \tfrac{3}{4} \text{ and } \dot{v} = 2\dot{y}(y + \tfrac{1}{2})$$

$$\therefore \frac{\dot{z}}{z} + \frac{2y\dot{y}}{y^2 + \frac{3}{4}} + \frac{\dot{y}}{y^2 + \frac{3}{4}} = 0 \therefore \text{L}z + \text{L}(y^2 + \tfrac{3}{4}) + \frac{2}{\sqrt{3}} \cdot \text{Arc, rad. } 1, \tan \frac{2y}{\sqrt{3}} = \text{const.}$$

$$\text{or } \text{L}z + \text{L}(v - \sqrt{v} + 1) = \frac{2}{\sqrt{3}} \cdot \text{Arc, rad } 1. \tan \frac{2}{\sqrt{3}} \cdot (v^{\frac{1}{2}} - \tfrac{1}{2}) = \text{C}$$

$$\text{or } \text{L}\left\{x - \sqrt{(xz)} + z\right\} + \frac{2}{\sqrt{3}} \cdot \text{Arc, tan} \frac{2}{\sqrt{3}} \left\{\left(\frac{x}{z}\right)^{\frac{1}{2}} - \tfrac{1}{2}\right\} = \text{C}$$

$$\text{and } \text{L}h + \frac{2}{\sqrt{3}} \cdot \text{Arc } 30^\circ = \text{C}$$

$$\therefore \text{L}\left(\frac{x - \sqrt{(xz)} + z}{h}\right) + \frac{2}{\sqrt{3}} \left\{\text{Arc, tan} \frac{2}{\sqrt{3}} \cdot \left\{\left(\frac{x}{z}\right)^{\frac{1}{2}} - \tfrac{1}{2}\right\} - \text{Arc } 30^\circ\right\} = 0;$$

let $z = 0$, $\therefore \text{L}\left(\frac{x}{h}\right) + \frac{2}{\sqrt{3}} \cdot \text{Arc } 60^\circ = 0$; assume $\frac{1}{\sqrt{3}} \cdot \text{Arc } 60^\circ = \theta$,

$\therefore \text{L}\left(\frac{x}{h}\right) + 2\theta = 0 \therefore x = h \cdot e^{-2\theta}$ when the upper cylinder is empty.

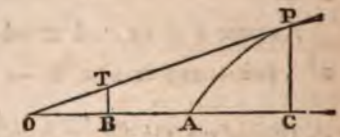
Now the time of emptying a cylinder, (rad $= r$) $= \frac{2\pi r^2}{a\sqrt{g}} \sqrt{\text{alt.}}$

$\therefore$ The whole time required $= \frac{2\pi r^2}{a} \sqrt{\frac{h}{g}} \cdot \left(1 + e^{-\theta}\right).$

XIV. QUESTION 384, *by* PALABA.

Find the equation of the curve of which this is the property; if from a fixed point in the axis a perpendicular be drawn to it, and produced to meet a tangent to any point in the curve, the length of this perpendicular and tangent together, shall be double the length of the curve between the vertex and the point from which the tangent was drawn.

FIRST SOLUTION, *by* PALABA, *the Proposer.*



Constructing the figure, let BA $= a$, BC $= x$, CP $= y$, AP $= z$; then BO $= \frac{y\dot{x}}{\dot{y}} - x$; therefore, from similar triangles, BT $= y - \frac{x\dot{y}}{\dot{x}}$.

Again, from sim. $\triangle$s, $\frac{y\dot{x}}{\dot{y}} : x :: \frac{y\dot{z}}{\dot{y}} :$ PT $= \frac{x\dot{z}}{\dot{x}}$.

$\therefore y - \frac{x\dot{y}}{\dot{x}} + \frac{x\dot{z}}{\dot{x}} = 2z$; take fluxions making $\dot{x}$ constant,

$\therefore -\frac{x\ddot{y}}{x} + \dot{z} + \frac{x\ddot{z}}{\dot{x}} = 2\dot{z} \therefore -\frac{x\ddot{y}}{\dot{x}} + \frac{x\ddot{z}}{\dot{x}} = \dot{z}$. But $\dot{z} = \sqrt{(\dot{x}^2 + \dot{y}^2)}$

$\therefore -\frac{x\ddot{y}}{\dot{x}} + \frac{x\dot{y}\ddot{y}}{\dot{x}\sqrt{(\dot{x}^2+\dot{y}^2)}} = \sqrt{(\dot{x}^2+\dot{y}^2)}$. Assume $\dot{y} = p\dot{x} \therefore \ddot{y} = \dot{p}\dot{x}$

$\therefore -x\dot{p} + \frac{xp\dot{p}}{\sqrt{(1+p^2)}} = \dot{x}\sqrt{(1+p^2)} \therefore \frac{\dot{x}}{x} - \frac{p\dot{p}}{1+p^2} + \frac{\dot{p}}{\sqrt{(1+p^2)}} = 0$;

$\therefore$ L $. x +$ L $. (p + \sqrt{(1+p^2)}) -$ L $. \sqrt{(1+p^2)} = c$,

or L $. x +$ L $. \left(1 + \frac{1}{\sqrt{\left(\frac{1}{p^2}+1\right)}}\right) = c$: now $\frac{1}{p} = \frac{\dot{x}}{\dot{y}} =$ tan $\angle$ OPC $= 0$, when $x = a$, $\therefore$ L $. a +$ L $. 2 = c \therefore c =$ L $. 2a$.

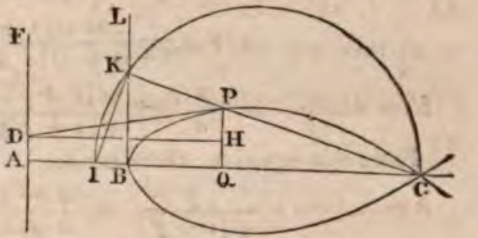
$\therefore$ L $. x +$ L $. \left(1 + \frac{1}{\sqrt{\left(\frac{1}{p^2}+1\right)}}\right) =$ L $. 2a \therefore 1 + \frac{1}{\sqrt{\left(\frac{1}{p^2}+1\right)}} = \frac{2a}{x}$ and $\frac{1}{\sqrt{\left(\frac{1}{p^2}+1\right)}} = \frac{2a-x}{x} \therefore \frac{1}{p^2} + 1 = \frac{x^2}{(2a-x)^2}$

$\therefore \frac{1}{p^2} = \frac{4a\,.\,(x-a)}{(2a-x)^2} \therefore p = \frac{1}{2\sqrt{a}} \cdot \frac{2a-x}{\sqrt{(x-a)}} \therefore \dot{y} = \frac{1}{2\sqrt{a}}$
$\cdot \frac{2a\dot{x} - x\dot{x}}{\sqrt{(x-a)}}$,

$\therefore y = \frac{1}{3\sqrt{a}} \cdot (4a - x) \cdot \sqrt{(x-a)}$, which wants no correction.

SECOND SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let BPC be the curve, B a point in which it meets the axis, A the fixed or given point in the axis, AF the perpendicular, and PD a tangent at any point P, meeting the perpendicular in D. Let PQ be any ordinate, and DH a perpendicular from D to PQ.



Put AB $= a$, BQ $= x$, PQ $= y$, arch BP $= z$.

By the theory of curves,

$dx : dy ::$ DH $= a + x :$ PH; also $dx : dz ::$ DH : DP.

Hence PH $= \frac{dy}{dx}(a+x)$, PD $= \frac{dz}{dx}(a+x)$, AD$=y - \frac{dy}{dx}(a+x)$.

Now, by the question, PD + AD $= 2z$; therefore

$$y + (a+x)\left(\frac{dz}{dx} - \frac{dy}{dx}\right) = 2z.$$

Put $p = \frac{dy}{dx}$ and $q = \frac{dz}{dx}$; and because $\frac{dz}{dx} = \sqrt{(1 + \frac{dy^2}{dx^2})}$, therefore $q = \sqrt{(1+p^2)}$; the letters p and q being now substituted in the foregoing equation, it becomes

$$y + (a+x)(q-p) = 2z \text{..............} (A)$$

Hence, taking the fluxions, we get,

$$dy + (a+x)(dq - dp) + (q-p)\,dx = 2dz.$$

In this equation, substitute pdx for dy and qdx for dz, then, after reduction, it becomes,

$$(a+x)(dq - dp) = qdx;$$

and hence, $\frac{dx}{a+x} = \frac{dq}{q} - \frac{dp}{q} = \frac{dq}{q} - \frac{dp}{\sqrt{(1+p^2)}}$.

This equation by integration gives

$$\log (a + x) = \log \surd(1 + p^2) - \log \left\{ p + \surd(1+p^2) \right\} + \log c;$$

and hence $a + x = \dfrac{c\surd(1+p^2)}{p + \surd(1 + p^2)}$ ....(B).

To determine the value of the arbitrary quantity c, we must recur to equation (A), and observing that x, y and z begin together, it there appears that when x, and consequently y and z are each $= 0$, then $a(q - p) = 0$, or $q = \surd(1 + p^2) = p$, hence, $1 + p^2 = p^2$, which shews, that when $x = 0$, then p is infinite: and in this case, equation (B) becomes $a = \dfrac{cp}{2p} = \dfrac{c}{2}$; hence $c = 2a$, and the adjusted equation is

$$a + x = \frac{2a\surd(1 + p^2)}{p + \surd(1 + p^2)}.$$

from which $$\frac{x}{a} = \frac{\surd(1+p^2) - p}{\surd(1+p^2) + p}.$$

From this equation, there results

$$\frac{p}{\surd(1 + p^2)} = \frac{a - x}{a + x},$$

and hence again,

$$p = \frac{a - x}{2\surd(ax)}$$

and since $p = \dfrac{dy}{dx}$ therefore

$$dy = \frac{dx\,(a - x)}{2\surd(ax)} = \frac{1}{2\surd a} \left\{ \frac{adx}{\surd x} - \surd x dx \right\}$$

and taking the fluents, so that x and y may begin together, we find

$$y = \frac{1}{2\surd a}\left(2a\surd x - \tfrac{2}{3}x^{\frac{3}{2}}\right);$$

and, after proper reduction,

$$9ay^2 = x\,(3a - x)^2.$$

This is the equation of the curve, which appears to be a parabolic line of the third order, the 68th species according to Newton.

To investigate a geometrical construction for any point in the curve, let us take $BC = 3BA = 3a$, then $CQ = 3a - x$, and the equation of the curve gives

$$3BC \times PQ^2 = BQ \times QC^2.$$

Draw CP, producing it to meet BL a perpendicular to BC in

K, and draw KI perpendicular to CK, meeting AB in I; then, by similar triangles and the property of a right-angled triangle,

$$CQ^2 : QP^2 :: CK^2 : KI^2 :: CB : BI.$$

But, by the equation of the curve,

$$CQ^2 : QP^2 :: 3BC : BQ :: BC : \tfrac{1}{3}BQ,$$

therefore $BQ = 3BI$, also $CQ = 3AI$. This property indicates an easy method of finding the point in which any ordinate PQ meets the curve, namely, by taking $BI = \frac{1}{3}BQ$, and describing a circle on IC as a diameter, to meet BL in K; a straight line drawn from C to K will manifestly pass through P the top of the ordinate.

We have found that $dy = \dfrac{dx(a-x)}{2\sqrt{(ax)}}$,

hence $dz = \sqrt{(dx^2 + dy^2)} = \dfrac{dx(a+x)}{2\sqrt{(ax)}}$,

and $dy + dz = \dfrac{adx}{\sqrt{(ax)}}$, therefore taking the fluents

$$y + z = 2\sqrt{(ax)}.$$

This formula indicates an elegant property of the curve.

XV. QUESTION 385, *by* PALABA.

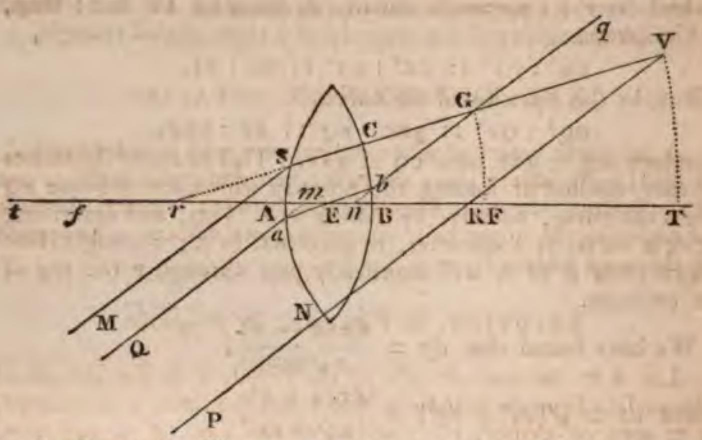
If the sine of incidence : sine of refraction :: $1 : n$, r and r' the radii of the surfaces, and t, its thickness, the distance (f) of the principal focus from the focal centre may be *accurately* determined from this expression,

$$\frac{1}{f} = \frac{1-n}{n} \cdot \left\{\frac{1}{r} + \frac{1}{r'} - \frac{1-n}{rr'} \cdot t\right\}.$$

Required the investigation.

SOLUTION, *by* PALABA, *the Proposer.*

Let AB be the lens whose axis is rR and centre E, R and r the centres of its surfaces, MS, Qa, PN a pencil of parallel rays incident upon it, of which Qabq passes through the centre; m and n the focal centres, also let PN be that ray which is incident perpendicularly on surface A. Take NV : RV :: $1 : n$, and V is the focus after the first refraction. Join Vr meeting the surface A in S and the line bq in G, then G is the focus of emergent rays (see Wood's Optics, 3rd Ed.). Then, from similar triangles, Rr : RV :: nr : nG the distance of the principal focus from



the focal centre. Then $\therefore \mathrm{NV} : \mathrm{RV} :: 1 : n$, $r : \mathrm{RV} :: 1 - n : n \therefore \mathrm{RV} = \frac{nr}{1-n}$, and from 1st propor. $r + r' - t : n \cdot \frac{r'}{1-n} :: \mathrm{E}n + \mathrm{E}r\ (nr) : n\mathrm{G} = \frac{n}{1-n} \cdot \frac{r}{r+r'-t} \cdot (\mathrm{E}n + \mathrm{E}r)$; now $\mathrm{EB} = \frac{tr'}{r+r'}$ (Wood's Optics) $\therefore \mathrm{E}r = r\mathrm{B} - \mathrm{EB} = r' - \frac{tr'}{r+r'} = \frac{r'}{r+r'} \cdot (r + r' - t)$. Also, if t, in the figure be the principal focus of parallel rays after the first refraction and incident on surface B, $\mathrm{E}t : \mathrm{E}r :: \mathrm{EB} : \mathrm{E}n$ (Wood's Optics), or $t\mathrm{B} - \mathrm{EB} : \mathrm{EB} :: \mathrm{E}r : \mathrm{E}n \therefore t\mathrm{B} - \mathrm{EB} : t\mathrm{B} :: \mathrm{E}r : \mathrm{E}n + \mathrm{E}r$, that is, $\frac{r'}{1-n} - \frac{tr'}{r+r'} : \frac{r'}{1-n} :: \frac{r'}{r+r'} \cdot (r + r' - t) : \mathrm{E}n + \mathrm{E}r = \frac{r+r'-t}{r+r'-(1-n)t} \cdot r'$, $\therefore n\mathrm{G}\ (f) = \frac{n}{1-n} \cdot \frac{rr'}{r+r'-(1-n)t}$

$$\therefore \frac{1}{f} = \frac{1-n}{n} \cdot \left\{ \frac{1}{r} + \frac{1}{r'} - \frac{1-n}{rr'} \cdot t \right\}. \qquad \textit{Q. E. D.}$$

Cor. If AB be a sphere, E is its centre, and m and n coincide with it, and $r = r' = \frac{t}{2} \therefore \frac{1}{f} = \frac{1-n}{n} \cdot \left\{ \frac{2}{r} - \frac{(1-n)2}{r} \right\} = \frac{1-n}{n} \cdot \frac{2n}{r} = \frac{(1-n)2}{r} \therefore f = \frac{r}{2(1-n)}$

which is the expression usually deduced for the focal length of a sphere.

XVI. QUESTION 386, *by* PALABA.

A prismatic vessel, of given dimensions, with its sides vertical, is filled with water; there are two given and equal orifices, one at the bottom, the other bisecting the altitude; required the time of emptying the upper half, supposing both orifices to be opened at the same instant?

SOLUTION, *by* PALABA, *the Proposer.*

Let $h =$ the semi-altitude of the vessel, $x =$ the altitude of the surface from the middle orifice when the time, t, is elapsed, $a =$ area of orifice, $r =$ radius of the base, $g = 32\frac{1}{6}$ feet. Then $a\sqrt{g} \cdot \sqrt{x}$ and $a\sqrt{g} \cdot \sqrt{(h + x)}$ are quantities respectively discharged per second through the middle and bottom orifices:

$$\therefore a\sqrt{g} \cdot (\sqrt{x} + \sqrt{(h+x)}) \cdot \dot{t} = -\pi r^2 \dot{x}$$

$$\therefore \dot{t} = -\frac{\pi r^2}{a\sqrt{g}} \cdot \frac{\dot{x}}{\sqrt{(h+x)}+\sqrt{x}} = -\frac{\pi r^2}{a\sqrt{g}} \cdot \frac{(h+x)^{\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x}}{h}$$

$$= \frac{\pi r^2}{ah\sqrt{g}} \cdot \left\{x^{\frac{1}{2}}\dot{x} + (h+x)^{\frac{1}{2}}\dot{x}\right\}$$

$$\therefore t = \frac{2\pi r^2}{3ah\sqrt{g}} \cdot \left\{x^{\frac{3}{2}} - (h+x)^{\frac{3}{2}}\right\} + \text{Cor.}$$

$$0 = \frac{2\pi r^2}{3ah\sqrt{g}} \cdot h^{\frac{3}{2}} \cdot \left\{1 - 2\sqrt{2}\right\} + \text{Cor.}$$

$$\therefore t = \frac{2\pi r^2}{3ah\sqrt{g}} \cdot \left\{x^{\frac{3}{2}} - (h+x)^{\frac{3}{2}}\right\} - \frac{2\pi r^2}{3a}\sqrt{\frac{h}{g}} \cdot (1 - 2\sqrt{2})$$

$\therefore$ The time required $= \dfrac{4\pi r^2}{3a}\sqrt{\dfrac{h}{g}} \cdot (\sqrt{2} - 1)$.

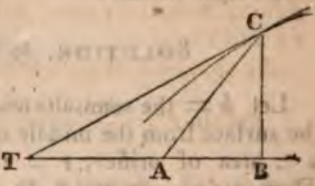
XVII. QUESTION 387, *by* PALABA.

TB, BC are the subtangent and ordinate of a curve whose vertex is A, and the tangent of the angle TCA is to the tangent of the angle ACB in a given ratio. What is the nature of the curve?

SOLUTION, *by* PALABA, *the Proposer.*

Assume $AB = x$, $BC = y$ $\therefore$ $BT = \frac{y\dot{x}}{\dot{y}}$ and $y : \frac{y\dot{x}}{\dot{y}} :: 1 : \tan TCB = \frac{\dot{x}}{\dot{y}}$; similarly $\tan ACB = \frac{x}{y}$ $\therefore$ $\tan TCA = \dfrac{\frac{\dot{x}}{\dot{y}} - \frac{x}{y}}{1 + \frac{\dot{x}}{\dot{y}} - \frac{x}{y}}$.

From the nature of the question



$$\frac{\frac{\dot{x}}{\dot{y}} - \frac{x}{y}}{1 + \frac{\dot{x}}{\dot{y}} - \frac{x}{y}} : \frac{x}{y} :: 1 : n$$

$$\therefore \frac{y\dot{x} - x\dot{y}}{x\dot{x} + y\dot{y}} = \frac{x}{ny},$$

$$\text{or } ny^2\dot{x} - nxy\dot{y} = x^2\dot{x} + xy\dot{y},$$

$$\text{or } ny^2\dot{x} = x^2\dot{x} + (n+1)\,xy\dot{y}.$$

Assume $y = vx$ $\therefore$ $\dot{y} = v\dot{x} + x\dot{v}$ $\therefore$ by substitution

$$nv^2x^2\dot{x} = x^2\dot{x} + (n+1)\,.\,vx^2\,.\,(v\dot{x} + x\dot{v})$$

$$\text{or } nv^2\dot{x} = \dot{x} + (n+1)\,(v^2\dot{x} + xv\dot{v})$$

$$\therefore \dot{x} + v^2\dot{x} + (n+1)\,xv\dot{v} = 0, \text{ or } \dot{x}(1+v^2) + (n+1)\,xv\dot{v} = 0$$

$$\therefore \frac{\dot{x}}{x} + (n+1)\frac{v\dot{v}}{1+v^2} = 0 \therefore Lx + \frac{n+1}{2}L(1+v^2) = Lc,$$

$$\text{or } Lx\,.\,(1+v^2)^{\frac{n+1}{2}} = Lc \therefore x\,.\,(1+v^2)^{\frac{n+1}{2}} = c$$

$$\therefore 1 + v^2 = \frac{c^{\frac{2}{n+1}}}{x^{\frac{2}{n+1}}} \therefore \frac{y^2}{x^2} = \frac{c^{\frac{2}{n+1}}}{x^{\frac{2}{n+1}}} - 1$$

$$\therefore y^2 = c^{\frac{2}{n+1}}\,x^{\frac{2n}{n+1}} - x^2 = (cx^n)^{\frac{2}{n+1}} - x^2.$$

If $n = 1$, the curve is a circle, which we know to be the case from the principles of geometry.

XVIII. QUESTION 388, *by Mr.* T. S. EVANS.

The most expeditious method of determining the latitude appears to be, by observing a number of altitudes near the meridian, with a repeating circle: and the following simple formula reduces them with great facility to the meridian altitude;

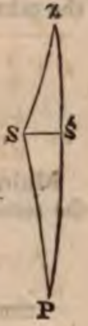
$$\sin \dot{\overline{\text{zs}}} = \text{ver sin P} \times \frac{\sin \text{PS} \sin \text{PZ}}{\sin \text{ZS}}.$$

Required its investigation?

SOLUTION, *by* L.

Let ss′ be the parallel of declination; then zs′ is the meridional zenith distance, and is equal to the difference between PZ and PS.

By spherical trigonometry (Simpson's Trig. p. 27.)



$$\text{ver sin P} = \frac{2 \sin \frac{1}{2}(\text{zs} + \text{zs}') \sin \frac{1}{2}(\text{zs} - \text{zs}')}{\sin \text{PS} \sin \text{PZ}},$$

therefore $2\sin \frac{1}{2}(\text{zs} - \text{zs}') = \dfrac{\text{versin P} \times \sin \text{PS} \sin \text{PZ}}{\sin \frac{1}{2}(\text{zs} + \text{zs}')}$.

But the observations being made when the object is near the meridian, the difference of the zenith distances zs and zs′ must necessarily be small; therefore, instead of $2 \sin \frac{1}{2}(\text{zs} - \text{zs}')$ we may put $\sin(\text{zs} - \text{zs}')$, and instead of $\sin \frac{1}{2}(\text{zs} + \text{zs}')$ the sine of zs; the above expression will then become

$\sin(\text{zs} - \text{zs}') = \dfrac{\text{ver sin P} \times \sin \text{PS} \sin \text{PZ}}{\sin \text{ZS}}$, which is the formula in the question; or the sines of small arcs being nearly equal to the arcs themselves, we may, instead of the above, take

$$\text{zs} - \text{zs}' = \frac{\text{ver sin P} \times \sin \text{PS} \sin \text{PZ}}{\sin \text{ZS}}.$$

XIX. QUESTION 389, *by* G. V.

Let a be an arc of a circle of which the radius is unity, then

$$a^4 = 5 \times 48^2 \times \frac{3 + \cos a - 4 \cos \frac{1}{2}a}{237 - \cos a + 124 \cos \frac{1}{2}a} \text{ nearly.}$$

Required the proof?

SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let t denote the tangent of an arc A, t_1 the tangent of its half, t_2 the tangent of its fourth and in general t_n the tangent of its 2^nth part.

By the theory of sines

$$\frac{1}{t} = \frac{1}{2t_1} - \frac{t_1}{2}.$$

Hence $\frac{1}{t^2} = \frac{1}{(2t_1)^2} - \frac{1}{2} + \frac{t_1^2}{4}$,

$$\frac{1}{t^4} = \frac{1}{(2t_1)^4} - \frac{1}{(2t_1)^2} + \frac{3}{8} - \frac{t_1^2}{4} + \frac{t_1^4}{16}.$$

When t_1 is small, the terms $\frac{t_1^2}{4}$, $\frac{t_1^4}{16}$ are small in respect of the others, and then

$$\frac{1}{t^2} = \frac{1}{(2t_1)^2} - \frac{1}{2} \text{ nearly},$$

$$\frac{1}{t^4} = \frac{1}{(2t_1)^4} - \frac{1}{(2t_1)^2} + \frac{3}{8} \text{ nearly}.$$

Multiply the sides of the first of these equations by 4 and the second by 3 and add the results, we then have

$$\frac{3 + 4t^2}{t^4} = \frac{3 + 4t_1^2}{(2t_1)^4} - \frac{14}{16};$$

similarly,

$$\frac{3 + 4t_1^2}{(2t_1)^4} = \frac{3 + 4t_2^2}{(4t_2)^4} - \frac{14}{16^2},$$

$$\frac{3 + 4t_2^2}{(4t_2)^4} = \frac{3 + 4t_3^2}{(8t_3)^4} - \frac{14}{16^3},$$

and so on indefinitely. Therefore adding the corresponding sides of these equations and rejecting what is common to the two sums, we get

$$\frac{3 + 4t^2}{t^4} = \frac{3 + 4t_n^2}{(2^n t_n)^4} - 14\left\{\frac{1}{16} + \frac{1}{16^2} + \frac{1}{16^3} \cdots\cdots + \frac{1}{16^n}\right\}.$$

Suppose now n indefinitely great; in this case $t_n = 0$, $2^n t_n = 2^n \tan \frac{A}{2^n} = A$, and the sum of the series, which will now consist of an infinite number of terms, is $\frac{14}{5}$, we have then

$$\frac{3 + 4t^2}{t^4} = \frac{3}{A^4} - \frac{14}{15},$$

$$\text{and } \frac{3}{A^4} = \frac{3 + 4t^2}{t^4} + \frac{14}{15}.$$

Now $t^2 = \tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$, therefore

$$\frac{3 + 4t^2}{t^4} = \frac{7 + 6 \cos 2A - \cos^2 2A}{1 - 2 \cos 2A + \cos^2 2A}.$$

Hence, and by substituting $\frac{1 + \cos 4A}{2}$ for $\cos^2 2A$, we have

$$\frac{3}{A^4} = \frac{13 - \cos 4A + 12 \cos 2A}{3 + \cos 4A - 4 \cos 2A} + \frac{14}{15}$$

$$= \frac{237 - \cos 4A + 124 \cos 2A}{15(3 + \cos 4A - 4 \cos 2A)},$$

$$\text{and } A^4 = \frac{45(3 + \cos 4A - 4 \cos 2A)}{237 - \cos 4A + 124 \cos 2A}.$$

Now put $\frac{a}{4}$ for A, and we at last get

$$a^4 = 16^2 \times 3^2 \times 5 \times \frac{3 + \cos a - 4 \cos \frac{1}{2}a}{237 - \cos a + 124 \cos \frac{1}{2}a},$$

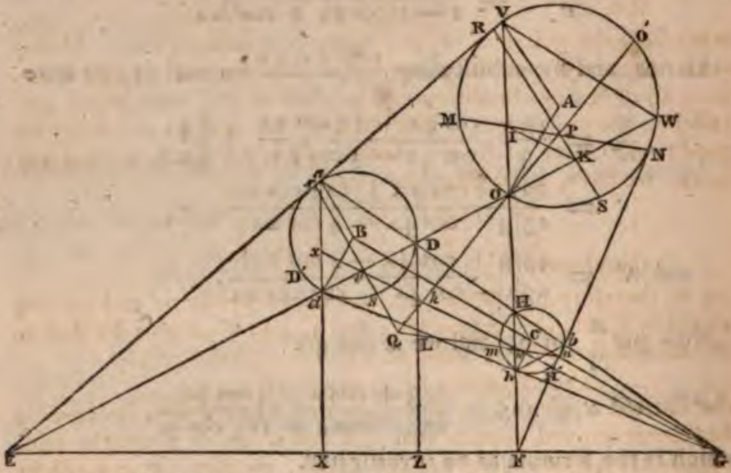
which is the formula to be investigated.

XX. PRIZE QUESTION 390, *by* M. G——r.

Three circles being given on the same plane; if two exterior tangents be drawn to one of them and each of the other two, the line joining the intersections of the chords of contact will meet the first circle in two points, which are the points of contact of this circle with two other circles, one of which touches the three given circles externally, and the other internally. Required the demonstration?

Solution, *by Mr.* Lowry, *R. M. College.*

Let A, B and C be the centres of the three circles given by position, and let these letters also represent the circles themselves. Let two tangents be drawn externally to the circles A and B to meet in E, and two others to the circles A and C to meet in F: Let RS and *rs* be the chords of contact of the first two, MN and *mn* those of the latter; Let RS and MN intersect in P, and *rs* and *mn* in Q, and let PQ be drawn to meet the circle A in O and O′: The proposition affirms that O is the point of contact of the circle A with another circle which is touched externally by the three circles A, B and C; and that O′ is the point of contact of the circle A with another circle which is touched internally by the three circles A, B and C. The fact is, that if OE and OF be drawn to meet the circles B and C in D and H respectively, the circle described through the three points O, D



and H will be touched at once, externally, by the three circles A, B and C in these points: or if O′E and O′F be drawn to meet the circles B and C in D′ and H′, the circle described through the three points O′, D′ and H′ will be touched internally by the three circles A, B and C in these points.

To prove this, let OE meet the circles A and B again in W and *d*, and *rs* in *v*; also let OF meet A and C in V and *h*, and *mn* in *w*. Draw *da* parallel to OV and *hb* to OW and join *a*D, H*b* and DH. It is a well known property of the point E that any straight line drawn from it to meet the circles B and A, cuts off similar segments from these circles; therefore the segment *da*D is similar to the segment OVW; wherefore the angle *da*D is equal to the angle OVW; and *ad*D is equal to VOW because *ad* is parallel to VO, therefore *a*D*d* is equal to VWO and consequently *a*D is parallel to VW.

In the same manner it may be proved that H*b* is parallel to VW, because FV cuts off similar segments from the circles C and A. All therefore that is now necessary to be proved is, that DH is on the same straight line with *a*D and H*b*; for the lines VW, *ad* and *hb* will then be parallel to DH, OH and OD, the lines which join the points of contact, and consequently a circle described through O, D and H will touch in these points the three circles A, B and C. Let two tangents be drawn to the circles B and C to meet in G; then it is a well known property that the three points E, F and G are in a straight line, and that BC if produced will pass through G. Draw the radii Bd, Ba, Ch and CH. Because of the similar segments daD, OVW; hbH and OWV, Bd and Ch are each parallel to the radius OA; consequently Bd is parallel to Ch.

$$\text{But } GC : GB :: \text{rad. } C : \text{rad. } B :: Ch : Bd;$$

wherefore the points d, h and G are in a straight line. In the same manner it may be proved that the points a, H and G are in a straight line, because aB is parallel to HC, each being parallel to the radius VA. Join IK and vw, and let the latter be produced to meet ad in x. Because vQ is parallel to PK and wQ to IP, we have, by similar triangles,

$$vO : OQ :: OK : OP,$$

$$OQ : Ow :: OP : OI;$$

$$\text{therefore } vO : Ow :: OK : OI.$$

vw is therefore parallel to IK, and consequently ad and OV are similarly divided in the points x and I, and Hh is also divided in the same manner in w, because $VI : OI :: Hw : wh$;

$$\text{wherefore } ax : xd :: Hw : wh$$

therefore the points v, w and G are in a straight line because it has been proved already that d, h and G are in a straight line. Again let DZ be drawn parallel to OF, meeting vw in k, dh in L, and EF in Z, and produce ad to meet EF in X. Because aX is parallel to VF and similarly divided, $ax : xd :: aX : dX$ ($:: HF : hF$); and by the property of tangents, DE is harmonically divided in the points E, d, v and D, therefore DZ, which meets the straight lines vG, dG and EG, is also harmonically divided, that is,

$$Dk : kL :: DZ : LZ;$$

and it has been shewn above, that

$$ax : xd :: aX : dX,$$

and by parallel lines

$$hL : LZ :: xd : dX;$$

$$\text{therefore } Dk : DZ :: ax : aX \; (:: HF : Hw);$$

wherefore, because EG and xG are straight lines, the points a, D, H, G are also in a straight line; consequently DH is parallel to VW.

In the same manner it may be proved that a circle described through the points O', D', H' will touch the three given circles.

Cor. When one of the circles as C vanishes to a point, the proposition in the question is still true: For if CG be drawn parallel to the chord MN to meet the chord rs in Q, the line PQ will pass through O, and a circle described through the points O, D, C will touch the circles in O and D; or if described through O', D' and C will touch them in O' and D'. Likewise if two of the circles as B and C be diminished to points, and

BQ and CQ be drawn parallel to the chords MN and VW (the points E and F then coalescing with B and C) the line PQ will cut the circle in O and O′, and a circle described through the points B and C and either of the points O, will touch the circle in that point. The demonstrations for these two particular cases are included in the general one.

SECOND SOLUTION. *Taken from the Annales des Mathematiques.*

Let c, c', c'' be the three given circles,
C the circle that touches all the three,
r, r', r'' the radii of the given circles,
R the radius of the touching circle,
t, t', t'' the points of contact,
a, a', a'' / b, b', b'' the co-ordinates of the centres of the given circles, the common origin being the centre of c'',
A, B the co-ordinates of the centre of the touching circle,
x, y the co-ordinates of the point t''.

To fix our ideas, we shall suppose all the contacts to be exterior, however, by a proper change in the signs, the investigation will suit every case.

From the position of the lines and the hypothesis we have just made, we get

$$x^2 + y^2 = r''^2 \dots\dots(1)$$

$$(A - a)^2 + (B - b)^2 = (R + r)^2 \dots\dots(2)$$

$$(A - a')^2 + (B - b')^2 = (R + r')^2 \dots\dots(3)$$

$$A^2 + B^2 = (R + r'')^2 \dots\dots(4)$$

$$\frac{x}{y} = \frac{A}{B} \text{ and } yA = Bx \dots\dots(5).$$

From equations (4, 5) we deduce

$$A = \frac{x(R + r'')}{\sqrt{(x^2 + y^2)}}, \qquad B = \frac{y(R + r'')}{\sqrt{(x^2 + y^2)}};$$

and from these again, by equation (1),

$$A = \frac{x(R + r'')}{r''}, \qquad B = \frac{y(R + r'')}{r''}.$$

By subtracting the sides of equations (2, 3) from the corresponding sides of equation (4) and putting the results under the form most convenient for eliminating R, we have

$$2aA + 2bB - 2(r'' - r)(R + r'') = a^2 + b^2 - (r'' - r)^2,$$

$$2aA + 2b'B - 2(r'' - r')(R + r'') = a'^2 + b'^2 - (r'' - r')^2.$$

By substituting in these the values of A and B they become

$$2\left\{ax+by-r''(r''-r)\right\}(R+r'')=r''\left\{a^2+b^2-(r''-r)^2\right\}$$

$$2\left\{a'x+b'y-r''(r''-r')\right\}(R+r'')=r''\left\{a'^2+b'^2-(r''-r')^2\right\}$$

Hence by eliminating $R+r''$ we have

$$\frac{ax+by-r''(r''-r)}{a^2+b^2-(r''-r)^2}=\frac{a'x+b'y-r''(r''-r')}{a'^2+b'^2-(r''-r')^2}\ \ldots\ldots(6)$$

This is the equation of a line that cuts the circle c'' in the point t'', and as it is only of the first degree it is evidently a straight line.

If we had supposed that the circle C had surrounded the three circles c, c', c'' the signs of r, r', r'' must have been changed into the opposite, but the equation of the co-ordinates of t'' would have been identical with that given above. Hence by the way we may infer that the two intersections of the line and the circle c'' are the two positions of t'' which correspond to the two cases of the circles c, c', c'' falling entirely without C, and entirely within it. There are eight cases in all.

The equation (6) will be satisfied if we make

$$ax+by=r''(r''-r)\ \ldots\ldots\ldots\ldots\ldots(7)$$

$$a'x+b'y=r''(r''-r')\ \ldots\ldots\ldots\ldots\ldots(8).$$

These equations therefore determine the co-ordinates of a point in the straight line expressed by equation (6), and if resolved, they will give the position of a point in that line. But each may also be regarded as the equation of a straight line given by position, and then the two will determine the co-ordinates of the point in which they cut each other which will also be a point in the line (6). As the equations have exactly the same form, it is evident that the line expressed by equation (7) has the same position in respect to the circles c'', c as the line expressed by equation (8) has in respect to the circles c'', c', that is, each line is determined from the two circles by precisely the same construction.

Equation (6) will also be satisfied, if we assume that the numerator of each side is equal to its denominator, this gives

$$a(x-a)+b(y-b)=r(r''-r)\ \ldots\ldots\ldots(9)$$

$$a'(x-a')+b'(y-b')=r'(r''-r)\ \ldots\ldots\ldots(10).$$

These equations determine the position of another point in the straight line expressed by equation (6). They also may be regarded as the equations of two straight lines which intersect in that point.

If, in equation (7), we change x and y into $a-x$ and $b-y$ respectively, and also permute between themselves the two radii r, r'', equation (7) will then become equation (9); a similar process will change equation (8) into equation (10), hence we may infer that the straight line (9) is related to the circle c, exactly as the straight line (7) is to the circle c'', and that the straight line (10) is related to c' in the very same manner: each of the four lines (7, 8, 9, 10) will therefore be determined by a like construction, so that we have only to enquire what is the position of some one of them, as (7).

Let x' and y' be the co-ordinates of any point in the circumference of c'', their relation is expressed by this equation

$$a'^2 + y'^2 = r''^2, \ldots\ldots\ldots\ldots (\alpha)$$

the equation of a tangent to the circle at that point is

$$xx' + yy' = r''^2 \text{ or } y = -\frac{x'}{y'}x + \frac{r''^2}{y'},$$

and this tangent is indeterminate, since x' and y' are connected only by equation (α). If, to determine the tangent, we suppose that it touches also the circle c, this condition, (which requires that a perpendicular drawn from its centre on the tangent be equal to the radius) will be expressed by the equation

$$\frac{b + \frac{ax'}{y'} - \frac{r''^2}{y'}}{\sqrt{\left(1 + \frac{x'^2}{y'^2}\right)}} = r, \text{ or } \frac{ax' + by' - r''^2}{\sqrt{(x'^2 + y'^2)}} = r,$$

or, having respect to equation (α)

$$ax' + by' = r''(r'' - r) \ldots\ldots\ldots\ldots (\beta)$$

By combining the equations (α, β) we shall have the co-ordinates of the point where the tangent common to c and c'' touches c'', and since equation (α) is of the second degree, it shews that there are two such tangents, and consequently two points of contact.

But instead of resolving equations (α, β) it will come to the same thing and be more convenient if we construct the two lines which they express, and these by their intersections will give the points sought; since then one of these (α) is the equation of the circle c'' itself, the other (β) which is of the first degree must be that of a straight line which joins the points where the circle is touched by the tangents common to it and the circle c: But equations (β) and (7) are manifestly the same, therefore (7) is the equation of the chord which joins the points in which c'' is touched by straight lines which touch also c: hence again it follows that (9) is the chord which joins the points in which the circle c is touched by the same lines: and similarly that (8) and (10) are the chords of contact with c'' and c' of tangents common to the two circles. Recollecting now that the point t'' is in a line which passes through the intersections of the lines (7, 8), and also through the intersection of the lines (9, 10), the truth of the theorem enunciated in the question is manifest.

NOTICES.

I. OBITUARY.

Gaspard Monge.

This celebrated French Mathematician, who now lives only in his writings, was born at Beaune, in 1746. He studied in the college of Lyons, and made such proficiency that at the early age of seventeen, he taught physics, which he had learnt himself only a year before. His skill, as a draftsman, led to his being admitted a designer and eléve in the school of Military Engineers at Mézieres, and it was here that he first conceived the idea of his Descriptive Geometry, a theory of great importance in every kind of architectural construction, but which it appears was reserved by its author until the institution of the Normal school, it being the practice of the engineer officers to conceal carefully from the artillerists the principles of their profession. His scientific labours led to his being appointed an Instructor in Mathematics, in aid to Nollet and Bossut, and in the sequel he was made Titular Professor. In 1780 he went to Paris, where he was united with Bossut, who was then Titular Professor of a course of Hydrodynamics, instituted at the Louvre by Turgot, and here he began to teach Analytic Geometry to La Croix, Gay-Vernon, and others; but he still concealed his Descriptive Geometry, because, as he told his pupils, he was not allowed to reveal this secret.

In 1783 he succeeded Bezout as examiner in the Marine, and he then entirely left the school of Mézieres, where he had been a professor eighteen years, at this period he composed a small work on Statics, for the use of the officers in that service. The revolution, which called into action so much talent, placed him at the head of the Marine as minister, and in quitting this office he gave great assistance in the creation of the means of defence which his country, assailed on all sides, then required.

About this time the Normal School was instituted, and here, united with La Grange and La Place in the course of mathematical instruction, he gave his lessons in Descriptive Geometry. This institution, however, had no duration, it was immediately suppressed, and was succeeded by the Central School of Public Labours, called afterwards the Polytechnique School, in the original conception and afterwards in the direction of which Monge took a distinguished part.

The limits within which this brief notice must be confined will not allow a detailed enumeration of all the patriotic and scientific labours of this distinguished mathematician, suffice it to say, that by his intimate acquaintance with the resources of the arts, and the principles of chemical and mechanical science, he rendered great assistance in the removal of the statues and paintings from Italy to Paris (we speak, of course, in praise of his ingenuity, without considering whether their removal was proper or otherwise).

He afterwards, together with Berthollet, accompanied Bonaparte into Egypt, where in the midst of labours and difficulties arising out of his situation, he found leisure to cultivate the abstract sciences. Here he was president of the Institute at Cairo, and the general-in-chief the vice-president, a thing very remarkable, considering how seldom it is that, men following the military profession are disposed to yield precedence to the cultivators of science.

On his return to France he resumed his labours in the Polytechnique School. The suppression of this institution in the year 1815, was a terrible stroke to the venerable Philosopher, who had considered it as the basis of his future fame. His name was soon afterwards erased from the list of the members of the Institute, and not replaced when that Institution was remodelled, although, from the terms of the *Charter*, it appears he was entitled to that honour. At last, feeling deeply this indignity offered to him in his old age, he died, in December, 1818, at the age of seventy.

Notwithstanding the harshness which the French government evinced towards this venerable Mathematician, his name and character are held in the highest estimation by the numerous pupils he had formed to the service of his country, and he has found an able eulogist in one of them, M. Charles Dupin, who, in an *Historical Essay on the Services and Scientific Labours of Gaspard Monge*, has done honour to his feelings, and has strewed flowers on the tomb of his illustrious preceptor. A number of those who had the advantage of his instruction in the Polytechnique School, have manifested their gratitude by subscribing for a monument to perpetuate his memory, and have requested Berthollet to superintend its erection.

The following is a list of his works which have been published separately.

1. Traité Elémentaire de Statique; 1 vol. in 8°, 1786.
2. Description de l'art de fabriquer les canons; 1 vol. in 4°, 1794.
3. Leçons de Géométrie descriptive, donnèes à l'Ecole Normale, publiées d'abord en feuilles, d'après les sténographes, 1795.

Les mêmes leçons, publiées en 1 vol. in 4°, 1799.

4. Feuilles d'analyse appliqueés à la géométrie; in 1 vol. in fol. 1795. The fourth edition (in 4to.) contains in addition, La construction de l'Equation des cordes vibrantes, des developpements sur l'integration des equations aux différentielles partielles, les caracteristiques, &c., 1809.

For a complete list of his writings, see *Dupin*'s work above mentioned.

II. THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON, FOR 1817, AND 1818.

The Mathematical Papers contained in the vol. for 1817, are—1. Two Papers on the parallax of the Fixed Stars; By John Pond, Esq. Astronomer Royal.—2. Description of a Thermometrical Barometer for Measuring Altitudes; By the Rev. Francis John Hyde Wollaston.—3. Observations on the Analogy which subsists between the Calculus of Functions and other Branches of Analysis; By Charles Babbage, Esq;—4. Of the Construction of Logarithmic Tables; By Thomas Knight, Esq.—5. Two general propositions on the Method of Differences; By Thomas Knight, Esq.—6. Note respecting the Demonstration of the Binomial Theorem inserted in the preceding volume of the Philosophical Transactions; By

VI. FOREIGN BOOKS.

Mémoire sur les Transcendantes Elliptiques; par M. George Bidone. In 4° de 20 pages, Turin 1817.

Elémens d'Algèbre; par M. Bourdon. In 8vo, de 600 pages, à Paris.

Deuxième supplément a la Théorie Analitique des Probabilités; par M. le Marquis de La Place. In 4°. de 48 pages, à Paris.

Second Supplément de la Géométrie Descriptive; par M. Hachette, à Paris.

Mécanique Physique ou Traité expérimental et raisonné du mouvement et de l'équilibre considèrès dans les corps solides; par Joseph Mollet. In 8°. de 500 pages, à Avignon.

Les Œuvres d'Euclide, en Grec, en Latin, et en Français, d'apres un Manuscrit très-ancien, qui était resté inconnu jusqu'a nos jours; par F. Peyrard. Tome III. 4°. à Paris.

Histoire de L'Astronomie du Moyen Age; par M. Delambre. Paris, 1819. In 4°.

Connaissance des Tems 1821.

De la structure des Vaisseaux Anglais, considérés dans ses derniers perfectionnements; par Charles Dupin, Correspondant de l'Institut de France, &c.

Elémens de Géométrie à trois dimensions; par M. Hachette. In 8°. de 200 pages, à Paris.

Essai Historique sur le Probleme des trois corps, ou Dissertation sur la Théorie du mouvement de la Lune et des Planètes, abstraction faite de leur Figure; par Alfred Gautier. In 4°. de 300 papes, à Paris.

Traité de Géometrie Descriptive; par M. Portier. In 8°. de 100 pages, á Paris.

Fundamenta Astronomiae pro anno MDCCLV. deducta ex Observationibus viri incomparablis James Bradeley in Specula Astronomica Grenovicensi per annos 1750—1762, institutis Auctore Frederico Wilhelmo Bessel, Acad. Berol. atque Petrop. sodali Institut Gallici Corresp. Regiomonti 1818. (Folio).

Tabula ad Expeditorem calculum Logarithmi Summae vel differentiæ duarum quantitatum per Logarithmos tantum datarum (Latin and German). Altonæ 1818.

Præcipuorum inde a Neutono Conatuum Compositionem Virium Demonstrandi Rescensio Auctore Carolo Jacobi. Gottingæ, 1818.

ARTICLE IV.

Solutions to Questions proposed in Number XV.

I. QUESTION 391, *by Mr.* JOHN BAINES, JUNR.

To find two rational fractions, either of which being added to the square of the other, shall make the same sum; and also that their difference shall be a biquadrate number.

SOLUTION, *by Master* JOHN MILL, *London,*
(Under thirteen years of age.)

Let them be x and $a^4 + x$. Then $x + (a^4 + x)^2 = x^2 + a^4 + x$, or $x^2 + x + 2a^4 x + a^8 = x^2 + x + a^4$, or $2a^4 x + a^8 = a^4$, or $x = \frac{1 - a^4}{2}$. Hence $a^4 + x = \frac{1 + a^4}{2}$. So that the two numbers are $\frac{1 + a^4}{2}$ and $\frac{1 - a^4}{2}$, where a is any rational fraction.

If $a = \frac{1}{2}$, the numbers are $\frac{17}{32}$ and $\frac{15}{32}$, and if $a = \frac{1}{3}$, the numbers are $\frac{40}{81}$ and $\frac{41}{81}$, &c.

A mistake was made in transcribing this question for the press, but we shall here insert the question as it was sent by the Proposer, together with his solution.

Question. To find two rational fractions, either of which being added to the *square root* of the other shall make the same sum; and also, that their difference shall be a biquadrate number.

Solution. Assume $\frac{x^2}{(x + y)^2}$ and $\frac{y^2}{(x + y)^2}$ for the two fractions, then $\frac{x^2}{(x + y)^2} + \sqrt{\frac{y^2}{(x + y)^2}} = \frac{y^2}{(x + y)^2} + \sqrt{\frac{x^2}{(x + y)^2}} = \frac{x^2 + xy + y^2}{(x + y)^2}$, the same sum; also $\frac{x^2}{(x + y)^2} - \frac{y^2}{(x + y)^2} = \frac{x^2 - y^2}{(x + y)^2}$ must be a biquadrate number; to effect which, let us first make it a square, and its root being made a square will render the expression a biquadrate. To do this, put $x = r^2 + s^2$, and $y = 2rs$, then $\frac{x^2 - y^2}{(x + y)^2} = \frac{r^6 - 2r^2 s^2 + s^4}{(r^4 + 2rs + s^2)^2} = \left(\frac{r^2 - s^2}{r^2 + 2rs + s^2}\right)^2 =$ a square, its root being $\frac{r^2 - s^2}{r^2 + 2rs + s^2}$:

but the denominator of this expression is a square, therefore it only remains to make the numerator, viz. $r^2 - s^2$, $=$ a square, which it will be when $r = t^2 + u^2$, and $s = 2tu$, where t and u may be taken at pleasure.

Ex. suppose $t = 2$, and $u = 1$, then $r = t^2 + u^2 = 5$, $s = 2tu = 4$, $x = r^2 + s^2 = 41$, and $y = 2rs = 40$; hence

$$\frac{x^2}{(x+y)^2} = \frac{1681}{6561}, \text{ and } \frac{y^2}{(x+y)^2} = \frac{1600}{6561},$$

are two fractions that will answer. For

$$\frac{1681}{6561} + \sqrt{\frac{1600}{6561}} = \frac{1600}{6561} + \sqrt{\frac{1681}{6561}} = \frac{4921}{6561}, \text{ and}$$

$$\frac{1681}{6561} - \frac{1600}{6561} = \frac{81}{6561} = \left(\frac{3}{9}\right)^4; \text{ \&c.}$$

II. QUESTION 392, *by the Rev. Mr.* L. EVANS, *Royal Military Academy, Woolwich.*

The ordnance clock at this institution, being placed in my observatory, for the intent of rating it, prior to its removal to Scotland, for an experimental purpose of ascertaining the true figure of the earth, I adjusted the length of its pendulum precisely to beat seconds, sidereal time. I would now request to know, how many seconds per day, will the clock gain or lose, in the latitude of 57° 5′ north, being that of Aberdeen, upon the hypothesis of Newton's consideration of the equatorial diameter being to the polar as 230 to 229?

SOLUTION, *by the Rev. Mr.* L. EVANS, *the Proposer.*

The centrifugal force, produced by the rotatory motion of the earth round its axis, causes a variation in the gravity of bodies, in different latitudes. This force is opposed to the force of gravity, being greatest at the equator and least at the poles. And according to Sir Isaac Newton, the force of the former to that of the latter, is as 229 to 230. Vide the Principia, B. III. Prop. 19 and 20.

Now, if $l =$ length of a pendulum that vibrates seconds at the equator.

$l' =$ ditto at any latitude λ,

then $l' = l \times \left(1 + \frac{\sin^2 \lambda}{230}\right)$. Vide Playfair's Outlines of Natural Philosophy. VOL. II. P. 299.

Since the length of the pendulum, which vibrates seconds, at my observatory, latitude 51° 29′ 7″·6, is $39\frac{1}{8}$ inches, or nearly so, we shall have, by substitution, from this assumption. $39\frac{1}{8} = l \times \left(1 + \frac{\sin^2 \text{of } 51^\circ\, 29'\, 7''{\cdot}6}{230}\right)$; hence $l = 39{\cdot}02113$, the length of the pendulum that vibrates seconds at the equator. Having the length of this pendulum, we may now find the length of the pendulum that vibrates seconds, at the latitude of Aberdeen, viz. 57° 5′ north; for, as was before observed,

$$l' = l \times \left(1 + \frac{\sin^2 \lambda}{230}\right) = 39{\cdot}02113 \times \left(1 + \frac{\text{sine}^2 \text{ of } 57''\, 5'}{230}\right)$$

$= 39{\cdot}14068$ inches. And, by the last cited author's Outlines of Nat. Philosophy, vol. i. p. 133, we have $n = \sqrt{\frac{140850}{39{\cdot}14068}} = 59{\cdot}98797$ the number of vibrations this pendulum makes in a minute. Then $60 - 59{\cdot}98797 = 0''{\cdot}01203$ gained, in one minute, by the pendulum that vibrates seconds in lat. 51° 29′ 7″·6, when removed to lat. 57° 5′. And $0{\cdot}01203 \times 60 \times 24 = 17''{\cdot}3232$, the gain in 24 hours.

If, with Emerson, we take his length of the pendulum, at the equator, and for lat. 51° 30′, $\doteq 39{\cdot}027$ and $39{\cdot}1308$ respectively, the preceding formula will be

$$l' = 39{\cdot}027 \times \left(1 + \frac{\sin^2 \text{ of } 57^\circ\, 5'}{230}\right) = 39{\cdot}15174 \text{ inches.}$$

And $n = \sqrt{\frac{60^2 \times 39{\cdot}1308}{39{\cdot}15174}} = 59{\cdot}98338$; then $60 - 59{\cdot}98338 = {\cdot}01661$; and ${\cdot}01661 \times 60 \times 24 = 23''{\cdot}9184$, the gain in 24 hours, according to Emerson's numbers. Or, which is the same thing. $\sqrt{39{\cdot}1308} : \sqrt{39{\cdot}15174} :: 86400''$ in $24^h : 86423{\cdot}92$ nearly; then $86423{\cdot}92 - 86400 = 23{\cdot}92$, gain in 24 hours, taking the lat. = 51° 30′.

If we make use of M. Krafft's formula, we have $l' = 39{\cdot}0045 + 0{\cdot}206 \times \sin^2 \lambda$, in English inches. Vide Dr. Hutton's Mathematical Dictionary, p. 404. 1st. Ed. Then the length of the pendulum that vibrates seconds in lat. 57° 5′, will be

$l' = 39{\cdot}0045 + 0{\cdot}206 \times \sin^2 \text{ of } 57^\circ\, 5' = 39{\cdot}14966$ and $l' = 39{\cdot}0045 + 0{\cdot}206 \times \sin^2 \text{ of } 51^\circ\, 29'\, 7''{\cdot}6 = 39{\cdot}13062$, the length of the pendulum which vibrates seconds at my observatory.

Then as $\sqrt{39{\cdot}13062} : \sqrt{39{\cdot}14966} :: 86400 : 86421$, and $86421 - 86400 = 21''$, the gain in the 24 hours, according to Krafft's formula.

3. Again; if l = length of the pendulum, which vibrates seconds at the equator, as before denoted,
l' = ditto at any latitude λ
and d = difference of lengths of the pendulums at equator and pole,
then $l' = l + d \times \sin^2$ of λ. Now, if we take $l = 39{\cdot}0265$ and $d = {\cdot}1608$, we have
$l = 39{\cdot}0265 + {\cdot}1608 \times \sin^2$ of λ, for another formula.

If we apply this formula to our present caſe, we have $l' = 39{\cdot}0265 + {\cdot}1608 \times \sin^2$ of $57^\circ\ 5' = 39{\cdot}0265 + {\cdot}1133151 = 39{\cdot}1398151$, the length of the pendulum vibrating seconds in lat. 57° 5′.

And $l' = 39{\cdot}0265 + {\cdot}1608 \times \sin^2$ of $51^\circ\ 29'\ 7''{\cdot}6 = 39{\cdot}0265 + {\cdot}098446 = 39{\cdot}124946$ the length of the pendulum vibrating seconds, according to this hypotheſis, at my obſervatory.

Then $\sqrt{39{\cdot}124946} : \sqrt{39{\cdot}1398151} :: 86400 : 86416{\cdot}4$; and $86416{\cdot}4 - 86400 = 16''{\cdot}4$ the gain in 24 hours.

By aſſuming the lengths of the pendulums, vibrating seconds, at the equator and the pole, from Sir Iſaac Newton's Table of the length of pendulums, found in his Principia, Lib. III. Prop. XX. namely, 39·03135 and 39·20179 English inches, the formula becomes $l' = 39{\cdot}03135 + {\cdot}17044 \times \sin^2$ of $57^\circ\ 5' = 39{\cdot}03135 + {\cdot}1201084 = 39{\cdot}1514584$ the length of the pendulum vibrating seconds, in lat. 57° 5′.

And $l' = 39{\cdot}03135 + {\cdot}17044 \times \sin^2$ of $51^\circ\ 29'\ 7''{\cdot}6 = 39{\cdot}03135 + {\cdot}1043481 = 39{\cdot}1356981$.

Then $\sqrt{39{\cdot}1356981} : \sqrt{39{\cdot}1514584} :: 86400 : 86417{\cdot}4$ and $86417{\cdot}4 - 86400 = 17''{\cdot}4$, the gain in 24 hours.

If we proceed to use the before-quoted table of Sir Iſaac Newton, agreeably to the method of interpolation, the proceſs will be as follows:

Length of pendulum in lat. 60° is 36·74225 French inches

$$\begin{array}{lll} \text{Ditto} \ldots\ldots \text{in lat. } 55 & 36{\cdot}72966 & \left.\begin{array}{l} {\cdot}01259 \\ {\cdot}0135 \end{array}\right\} \text{differences} \\ \text{Ditto} \ldots\ldots \text{in lat. } 50 & 36{\cdot}71616 & \end{array}$$

Then, $5^\circ : {\cdot}01259 :: 2^\circ\ 5' : {\cdot}00525$

$$\begin{array}{r} 36{\cdot}72966 \\ \hline 36{\cdot}73491 \end{array}$$

36·73491 French inches, the length of the pendulum that vibrates seconds, in lat. 57° 5′.

Again, $5^\circ : {\cdot}0135 :: 1^\circ\ 29'\ 7''{\cdot}6 : {\cdot}00401$

$$\begin{array}{r} 36{\cdot}71616 \\ \hline 36{\cdot}72017 \end{array}$$

36·72017 French inches, the length of the pendulum that vibrates seconds in lat. 51° 29′ 7″·6, then $\sqrt{36{\cdot}72017} : \sqrt{36{\cdot}73491} :: 86400 : 86417{\cdot}32$, and

$86417{\cdot}32 - 86400 = 17''{\cdot}32$, the gain in 24 hours by this method.

Taking, moreover, Emerson's lengths of pendulums at the equator and pole, namely, 39·027 and 39·197, and using the last formula, we have $d = {\cdot}17$ and $l' = 39{\cdot}027 + {\cdot}17 \times \sin^2$ of $57^\circ\ 5' = 39{\cdot}027 + {\cdot}119798 = 39{\cdot}146798$, the length, of the pendulum vibrating seconds, in lat. 57° 5′.

And $l' = 39{\cdot}027 + {\cdot}17 \times \sin^2$ of $51^\circ\ 29'\ 7''{\cdot}6 = 39{\cdot}027 + {\cdot}10407 = 39{\cdot}13107$, the length of the pendulum vibrating seconds, in lat. 51° 29′ 7″·6. Then, as $\sqrt{39{\cdot}13107} : \sqrt{39{\cdot}146798} :: 86400 : 86417{\cdot}36$, and $86417{\cdot}36 - 86400 = 17''{\cdot}36$, the gain in 24 hours, using Emerson's table. Vide his Geography, Prop. XI.

I shall only make use of one table more, before I bring the subject to a conclusion, and that will be Maupertuis's. Vide his figure of the earth, or Martin's Institutions, art. 3391, who has reduced that table to English measure.

According to Maupertuis, the length of the pendulum vibrating seconds, at the equator, is 39·0754 and at the pole 39·2684, then we have, by the last formula, $l' = 39{\cdot}0754 + {\cdot}193 \times \sin^2$ of $57^\circ\ 5' = 39{\cdot}0754 + {\cdot}1360064 = 39{\cdot}2114064$ the length of the pendulum vibrating seconds, in lat. 57° 5′.

And $l' = 39{\cdot}0754 + {\cdot}193 \times \sin^2$ of $51^\circ\ 29'\ 7''{\cdot}6 = 39{\cdot}0754 + {\cdot}11816 = 39{\cdot}19356$, the same in lat. 51° 29′ 7″·6. Then, as $\sqrt{39{\cdot}19356} : \sqrt{39{\cdot}2114064} :: 86400 : 86419{\cdot}62$, and $86419{\cdot}62 - 86400 = 19''{\cdot}62$ gained in 24 hours, upon Maupertuis's hypothesis:

Lastly, let us make use of his table, by interpolation, and the operation will be thus;

Length of pendulum, in lat.	60°, is	39·2202	·0153	differences;
Ditto .... in	55	39·2049	·0146	
Ditto............... in	50	39·1903		

Then, as $5^\circ : {\cdot}0153 :: 2^\circ\ 5' : {\cdot}006375$

$$\begin{array}{r} {\cdot}006375 \\ \underline{39{\cdot}2049} \\ 39{\cdot}211275 \end{array}$$

the length of the pendulum vibrating seconds, in lat. 57° 5′. Again;

$5^\circ : {\cdot}0146 :: 1^\circ\ 29'\ 7''{\cdot}6$ or $1^\circ{\cdot}485\frac{4}{9} : {\cdot}0053375$

$$\begin{array}{r} {\cdot}0053375 \\ \underline{39{\cdot}1903} \\ 39{\cdot}1956375 \end{array}$$

the length of the pendulum vibrating seconds, in lat. 51° 29′ 7″·6. Then, as $\sqrt{39{\cdot}1956375} : \sqrt{39{\cdot}211275} :: 86400 : 86417{\cdot}22$ and $86417{\cdot}22 - 86400 = 17''{\cdot}22$ gained in 24 hours, by this mode of proceeding, differing 2″·4 from the former.

DIGEST OF THE WHOLE.

Order.	Length of pendulum vibrating in latitude 51° 29′ 7″·6.	Length of pendulum vibrating in lat. 57° 5′	Time gained in 24 hours.	The name of Author's formula, and table of lengths of Pendulums used.
			″	
1	39·125	39·14068	17·3232	Playfair's formula first length assumed
2	39·1308	39·15174	23 92	Do. and Emerson's Table.
3	39·13062	39 14966	21·	Krafft's Formula.
4	39·124946	39·1398151	16·4	The third Formula.
5	39·1356981	39·1514584	17·4	Do. and Sir Isaac Newton's Table.
6	36·72017 Fr. I.	36·73491 Fr. I.	17·32	By Interpolation of do.'s Table.
7	39·13107	39·146798	17·36	Emerson's Table and third Formula
8	39·19356	39·2114064	19·62	Maupertuis's Table and third Formula.
9	39·1956375	39·211275	17·22	By Interpolation of Maupertuis's Table.

167·5632 Aggregate of the gains in 24 hours.

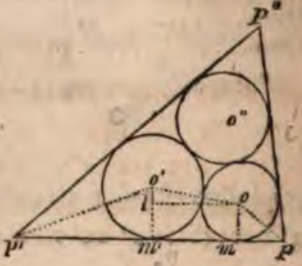
18·6181 The mean of the 9 results,

being the seconds gained in 24 hours, by the clock, if removed to lat. 57° 5′ north. The discrepancy, in the lengths of the pendulums, arises from the hypotheses of the several Authors, as to the proportion of the earth's equatorial and polar axes, and its equal density, a supposition which is not very probable.

III. QUESTION 393, *by* ITALICUS.

To describe three circles in a triangle, each of which may touch the other two, and also two sides of the triangle,

SOLUTION, *from the Annales de Mathématiques.*



Designate by p, p', p'' the angular points of the given triangle; by c, c', c'' the sides respectively opposite; by o, o', o'' the respective centres, and by r, r', r'' the radii of the circles required, and adopting the following abreviations.

$$s - c = \varphi, \qquad c'c''\varphi = sd^2,$$
$$c + c' + c'' = 2s,\ s' - c' = \varphi',\ \text{R}^2 s = \varphi\varphi'\varphi'',\ cc''\varphi' = sd'^2,$$
$$s - c'' = \varphi'', \qquad cc'\varphi'' = sd''^2;$$

then R is the radius of the inscribed circle; d, d', d'' the respective distances of its centre from the points p, p', p''; and $\varphi, \varphi', \varphi''$, the respective distances of these points from the points of contact of this circle with the sides of the triangle. From these equations we easily deduce the following relations:

$$\varphi'\varphi''d^2 = \text{R}^2c'c'' \qquad \varphi d'd'' = \text{R}cd,$$
$$\varphi+\varphi'+\varphi'' = s, \varphi\varphi''d'^2 = \text{R}^2cc'', \text{R}cc'c'' = sdd'd'', \varphi'dd'' = \text{R}c'd',$$
$$\varphi\varphi'd''^2 = \text{R}^2cc' \qquad \varphi''dd = \text{R}c''d''$$

From o and o' upon c'' let fall the perpendiculars $om = r$, $o'm' = r'$, join o,o' and from o parallel to c'' draw ol meeting $o'm'$ in l, the triangle olo' is right-angled at l, therefore

$$ol = c''-pm-p'm' = \sqrt{\left\{(r'+r)^2-(r'-r)^2\right\}} = 2\sqrt{(rr')};$$

But we have

$$pm = r\cot\tfrac{1}{2}p = r\sqrt{\frac{\varphi s}{\varphi'\varphi''}} = \frac{\varphi}{\text{R}}r, p'm' = r'\cot\tfrac{1}{2}p' = r'\sqrt{\frac{\varphi's}{\varphi\varphi'}} = \frac{\varphi'}{\text{R}}r';$$

substituting which, it becomes

$$c''-\frac{\varphi}{\text{R}}r-\frac{\varphi'}{\text{R}}r' = 2\sqrt{(rr')};$$

taking away the denominator, and transposing and forming analogous equations for the other circles, we have

$$\varphi r + 2\text{R}\sqrt{(rr')} + \varphi'r' = \text{R}c'',$$
$$\varphi r + 2\text{R}\sqrt{(rr'')} + \varphi''r'' = \text{R}c',$$
$$\varphi'r' + 2\text{R}\sqrt{(r'r'')} + \varphi''r'' = \text{R}c;$$

which are the equations of the problem.

If we suppose $r' = rx'^2$, $r'' = rx''^2$ the three equations become

$$r(\varphi + 2\text{R}x' + \varphi'x'^2) = \text{R}c'',$$
$$r(\varphi + 2\text{R}x'' + \varphi'x''^2) = \text{R}c',$$
$$r(\varphi'x' + 2\text{R}x'x'' + \varphi''x''^2) = \text{R}c;$$

the last gives

$$r = \frac{\text{R}c}{\varphi'x'^2 + 2\text{R}x'x'' + \varphi''x''^2};$$

this value being substituted in the two first, they become

$$(\text{A}') \qquad c(\varphi + 2\text{R}x + \varphi'x'^2) = c''(\varphi'x'^2 + 2\text{R}x'x'' + \varphi''x''^2),$$
$$(\text{A}'') \qquad c(\varphi + 2\text{R}x'' + \varphi''x''^2) = c'(\varphi'x'^2 + 2\text{R}x'x'' + \varphi''x''^2);$$

and from these two equations we have now to get the values of x' and x'' and substitute them in the value of r.

If we multiply equation (A′) by $\frac{c'\varphi\varphi''}{s}$, and equation (A″) by $\frac{c''\varphi\varphi'}{s}$; develope both, and put for $\varphi\varphi'\varphi''$ its equal R^2s and remark that

$$s(c-c'') = c(s-c'') - c''(s-c) = c\varphi'' - c''\varphi,$$

$$s(c - c') = c(s - c') - c'(s - c) = c\varphi' - c'\varphi,$$

$$\frac{c'c''\varphi}{s} = d^2, \quad \frac{cc''\varphi'}{s} = d'^2, \quad \frac{cc'\varphi''}{s} = d''^2,$$

they become

$$\left\{d(\mathrm{R}x' + \varphi''x'')\right\}^2 - \left\{d''(\mathrm{R}x' + \varphi)\right\}^2 = 0,$$
$$\left\{d(\mathrm{R}x'' + \varphi'x')\right\}^2 - \left\{d'(\mathrm{R}x'' + \varphi)\right\}^2 = 0;$$

therefore

$$d(\mathrm{R}x' + \varphi''x'') = d''(\mathrm{R}x' + \varphi),$$
$$d(\mathrm{R}x'' + \varphi'x') = d'(\mathrm{R}x'' + \varphi),$$

which give

$$x' = \varphi.\ \frac{d''(d - d')\mathrm{R} - \varphi''dd'}{(d-d')(d-d'')\mathrm{R}^2 - \varphi'\varphi''d^2} = \frac{\varphi}{\mathrm{R}}.\ d''.\frac{c'' - (d-d')}{c'c'' - (d-d')(d-d'')},$$

$$x'' = \varphi.\ \frac{d'(d-d'')\mathrm{R} - \varphi'dd''}{(d-d')(d-d'')\mathrm{R}^2 - \varphi'\varphi''d^2} = \frac{\varphi}{\mathrm{R}}.\ d'.\ \frac{c - (d-d'')}{c'c'' - (d-d')(d-d'')};$$

but it has been seen that

$$\varphi\varphi'd''^2 = \mathrm{R}^2cc', \quad \varphi\varphi''d'^2 = \mathrm{R}^2cc'', \quad \varphi d'd'' = \mathrm{R}cd,$$

therefore we get

$$\varphi'x'^2 = \varphi c.\ \frac{c'\left\{c'' - (d - d')\right\}^2}{\left\{c'c'' - (d-d')(d - d'')\right\}^2},$$

$$2\mathrm{R}x'x'' = \varphi c.\ \frac{2d\left\{c'' - (d - d')\right\}\left\{c' - (d - d'')\right\}}{\left\{c'c'' - (d - d')(d - d'')\right\}^2},$$

$$\varphi''x''^2 = \varphi c.\ \frac{c''\left\{c' - (d - d'')\right\}^2}{\left\{c'c'' - (d - d')(d - d'')\right\}^2};$$

these values being written in the expression found above for r, it becomes

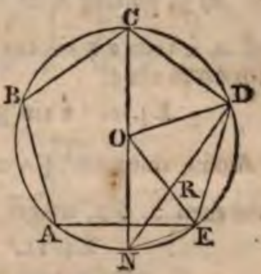
$$r = \frac{\mathrm{R}}{\varphi}.\ \frac{\left\{c'c'' - (d - d')(d - d'')\right\}^2}{c'\left\{c'' - (d-d')\right\}^2 + 2d\left\{c'' - (d-d')\right\}\left\{c' - (d-d'')\right\} + c''\left\{c' - (d-d'')\right\}^2}.$$

IV. QUESTION 394, *by* G V.

Prove, that if from the extremities of the side of a pentagon inscribed in a circle, straight lines be drawn to the middle of the arc, subtended by the adjacent side, their difference is equal to the radius: the sum of their squares is three times the square of the radius; and the rectangle contained by them is equal to the square of the radius.

First Solution, *by Mr.* Cunliffe, R. M. C.

Let ABCDE be a pentagon inscribed in a circle, the diameter NC being perpendicular to the middle of the side AE, and O the centre of the circle. Draw DO, EO, ND and NE: then ND and NE are straight lines from the extremities of one of the sides DE of the pentagon, to the middle of the arc subtended by the adjacent side AE. Let R be the intersection of DN and OE; the angle CND = DNE, therefore DN bisects the angle ONE: again, the angle NOE (NOR) = the angle CND (ONR), that is the angle NOR = ONR, and therefore NR = OR. Also the angle ORD = ERN = ONR + NOR = 2ONR = ONE = NER, that is, the angle ERN = NER, and therefore NE = NR; again, the angle COD = DOR = 2NOR = 2ONR = ONE = NER = ERN = ORD, that is, the angle DOR = ORD, and therefore DR = OD, and hence DN — NE = DN — NR = DR = OD, that is the difference of the lines DN and NE is equal to the radius of the circle.



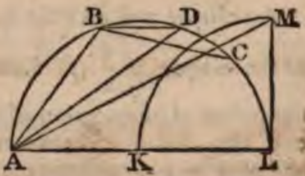
Per the similar isosceles triangles, NRO, NOD; ON = OD : OR = EN :: DN : OD, whence EN × DN = OD^2, that is the rectangle under the lines EN and DN is equal to the square of the radius.

We have shown that DN — EN = OD, wherefore $(DN - EN)^2 = DN^2 - 2\,EN \times DN + EN^2 = OD^2$; and we have proved that $2EN \times ND = 2\,OD^2$, and by addition $DN^2 + EN^2 = 3\,OD^2$.

Therefore all the properties mentioned in the question have been demonstrated.

Second Solution, *by Master* John Mill.

Let AB, BC, be two sides of a regular pentagon inscribed in a circle; bisect the arc BC in D, join AD and BD. Then since the arc AB or BC is one fifth of the whole circumference, BD is one fifth of the circumference of a semicircle, or two fifths of a quadrant. Hence AD, the sum of the arcs AB and BD, is six fifths of a quadrant. Hence the straight line BD is the chord of two fifths, or twice the sine



of one fifth : and AD is the chord of six fifths, or twice the sine of three fifths.

Put one fifth of a quadrant $= \varphi$. Then, since, if A and B are two arcs, $2 \sin A \times \sin B = R \times \left\{ (\cos (A - B) - \cos (A + B) \right\}$, putting $A = 3\varphi$ and $B = \varphi$, we get $2 \sin 3\varphi \sin \varphi = R \times (\cos 2\varphi - \cos 4\varphi)$, or, since $\cos 2\varphi = \sin 3\varphi$, and $\cos 4\varphi = \sin \varphi$,

Eq. I. $2 \sin 3\varphi \sin \varphi = R (\sin 3\varphi - \sin \varphi)$.

Again, since $\sin^2 \frac{1}{2} A = \frac{R (R - \cos A)}{2}$, putting A first $= 2\varphi$, and then $= 6\varphi$, we get

Eq. II. $2 \sin^2 \varphi = R (R - \sin 3\varphi)$, and

Eq. III. $2 \sin^2 3\varphi = R (R + \sin \varphi)$.

Subtracting eq. 2 from eq. 3, and dividing both sides by $\sin 3\varphi + \sin \varphi$, we get,

Eq. IV. $2 \sin 3\varphi - 2 \sin \varphi = R$.

But we have seen that $AD = 2 \sin 3\varphi$, and $BD = 2 \sin \varphi$; therefore $AD - BD = R$. *q. e. d.*

Again, $AD^2 + BD^2 = 4 (\sin^2 3\varphi + \sin^2 \varphi)$ or, by eq. 2 and 3, $= R \left\{ 4R - (2 \sin 3\varphi - 2 \sin \varphi) \right\} =$ (by eq. 4) $R \times (4R - R) = 3 R^2$. *q. e. d.*

Again, from eq. 1 and 4.

Eq. V. $4 \sin 3\varphi \sin \varphi = R^2$

or, $AD \times BD = R^2$. *q. e. d.*

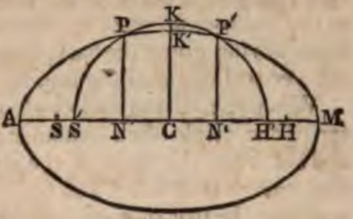
Scholium. Many other properties may be investigated in the same manner, for example that if AL be the diameter, and LM be taken on the tangent, equal to radius, the sum of AD and BD $= \frac{1}{2}$AM.

V. QUESTION 395, *by* MATH. PEMB. CANTAB.

An ellipse and a circle having the same centre, the axis major of the ellipse $= \frac{3}{2} \times$ diameter of the circle: To determine the points of intersection of the two curves, supposing the foci to cut the distance between the extremities of the diameter and axis in a given ratio; and to determine that ratio when the ellipse and circle just touch each other.

FIRST SOLUTION, *by* MATH. PEMB. CANTAB. *the Proposer.*

Let S and H be the foci of the ellipse, $S'H'$ the diameter of the circle; put $AS' = a$, $AS = \frac{a}{m}$, $AM = d$, $CN = x$, $NP = y$. Then $AS' = AC - S'C = \frac{d}{2} - \frac{d}{3}$, or $a = \frac{d}{6}$, $\therefore d = 6a$, $AC^2 = \frac{d^2}{4} = 9a^2$, $S'C^2 = \frac{d^2}{9} = 4a^2$, and $CK'^2 = AS.SM = \frac{a}{m}\left(d - \frac{a}{m}\right) = \frac{a^2}{m^2}(6m - 1)$.



But, by the property of the ellipse,

$$AN.NM \text{ or } AC^2 - CN^2 : NP^2 :: AC^2 : CK'^2,$$

that is $9a^2 - x^2 : y^2 :: 9a^2 : \frac{a^2}{m^2}(6m - 1) :: 9 : \frac{6m - 1}{m^2}$

$$\therefore y^2 = \frac{6m - 1}{9m^2} \times (9a^2 - x^2).$$

By the nature of the circle $NP^2 = PC^2 - NC^2 = S'C^2 - NC^2$, or $y^2 = 4a^2 - x^2 \therefore \frac{6m - 1}{9m^2} \times (9a^2 - x^2) = 4a^2 - x^2$,

$$\text{or } 54ma^2 - 9a^2 - 6mx^2 + x^2 = 36m^2 a^2 - 9m^2 x^2,$$

that is $(3m - 1)^2 \times x^2 = (4m^2 - 6m + 1) \times 9a^2$

$$\therefore x = \frac{3a}{3m - 1} \sqrt{(4m^2 - 6m + 1)}$$

$$= \frac{d}{6m - 2} \sqrt{(4m^2 - 6m + 1)}.$$

When they touch at K, $CK' = CK = S'C = \frac{d}{3} = 2a$ and $x = NC = 0$, $\therefore \frac{4a}{3m - 1} \sqrt{(4m^2 - 6m + 1)} = 0$, and $\therefore 4m^2 - 6m + 1 = 0$, and $m = \frac{3}{4} + \frac{1}{4}\sqrt{5} \therefore AS = \frac{4\,AS'}{3 + \sqrt{5}}$, or $AS : AS' :: 4 : 3 + \sqrt{5}$.

This may also be found otherwise, for since $CK' = S'C$, $\frac{a^2}{m^2}(6m - 1) = 4a^2 \therefore 6m - 1 = 4m^2$, or $4m^2 - 6m + 1 = 0$, the same as before.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

The distance of the foci from the centre of the ellipse will be easily found from the data, which obviously reduces the question to the following, viz. to that of determining a plane triangle, from having given the base, the sum of the sides, and the line from the vertical angle to the middle of the base.

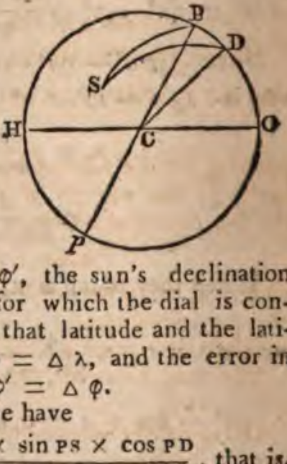
When the ellipse and circle just touch each other, the diameter of the circle then becomes the conjugate axis of the ellipse, and therefore the ratio of the transverse axis to the conjugate in that case is as 3 to 2; and the ratio mentioned in the question, will then be that of $3 - \sqrt{5}$ to $\sqrt{5} - 2$.

VI. QUESTION 396, *by* G. V.

A horizontal dial, constructed for a given latitude, is set up in a latitude a little different; what will be the error in the time it shews at a given hour on a given day?

SOLUTION, *by Mr.* JOHN WALLACE, *R. M. College.*

Let HPO*p* represent the meridian, P*p* the axis of the earth, CD the stile of the dial, S the sun, and SP, SD arcs of great circles of the sphere, passing through S, and through the points P and D. Then it is evident that HPS is the true horary angle, and HDS the angle measured by the dial. Hence the difference of the angles HPS and HDS is the error to be determined.



Put angle HPS $= \varphi$, and PDS $= \varphi'$, the sun's declination for the given time $= \alpha$, the latitude for which the dial is constructed $= \lambda$, the difference between that latitude and the latitude of the place where it is set up $= \Delta\lambda$, and the error in time shewn by the dial, that is, $\varphi - \varphi' = \Delta\varphi$.

Then, by spherical trigonometry, we have

$$\cot \text{D} = \frac{\cos \text{PS} \times \sin \text{PD} + \cos \text{HPS} \times \sin \text{PS} \times \cos \text{PD}}{\sin \text{HPS} \times \sin \text{PS}}, \text{ that is,}$$

$$\cot \varphi' = \frac{\sin \alpha \sin \Delta\lambda + \cos \varphi \cos \alpha \cos \Delta\lambda}{\sin \varphi \cos \alpha}. \text{ Therefore,}$$

since $\cot \varphi = \frac{\cos \varphi}{\sin \varphi}$, we obtain

$$\cot\varphi' - \cot\varphi = \frac{\sin\alpha\sin\Delta\lambda + \cos\varphi\cos\alpha\cos\Delta\lambda}{\sin\varphi\cos\alpha} - \frac{\cos\varphi}{\sin\varphi}.$$

But $\cot\varphi' - \cot\varphi = \frac{\sin(\varphi - \varphi')}{\sin\varphi\sin\varphi'} = \frac{\sin\Delta\varphi}{\sin\varphi\sin\varphi'}$; wherefore we have, by substitution and reduction,

$$\sin\Delta\varphi = \tan\alpha\sin\Delta\lambda\sin\varphi' + \cos\varphi\sin\varphi'(\cos\Delta\lambda - 1).$$

Now $\sin\varphi'\cos\varphi = \frac{1}{2}\sin(\varphi + \varphi') - \frac{1}{2}\sin(\varphi - \varphi')$, therefore the above equation becomes,

$$(1 + \cos\Delta\lambda)\sin\Delta\varphi = 2\tan\alpha\sin\Delta\lambda\sin\varphi' - (1 - \cos\Delta\lambda)\sin(\varphi + \varphi').$$

Hence, dividing by $(1 + \cos\Delta\lambda)$, and observing that

$$\frac{\sin\Delta\lambda}{1 + \cos\Delta\lambda} = \tan\tfrac{1}{2}\Delta\lambda, \text{ and } \frac{1 - \cos\Delta\lambda}{1 + \cos\Delta\lambda} = \tan^2\tfrac{1}{2}\Delta\lambda,$$

we find

$$\sin\Delta\varphi = \tan\tfrac{1}{2}\Delta\lambda\left\{2\tan\alpha\sin\varphi' - \tan\tfrac{1}{2}\Delta\lambda\sin(\varphi + \varphi')\right\}.$$

From this expression for $\sin\Delta\varphi$ we shall be able, if we suppose $\varphi = \varphi'$, to find an approximate value of $\Delta\varphi$, the error of the time indicated by the dial. But if we wish to find the exact value of $\Delta\varphi$ it will be necessary to eliminate φ from the above formula. For this purpose put $\varphi = \varphi' + \Delta\varphi$; then

$$\sin(\varphi + \varphi') = \sin(2\varphi' + \Delta\varphi) =$$
$$\sin 2\varphi'\cos\Delta\varphi + \cos 2\varphi'\sin\Delta\varphi.$$

Hence, by substitution, we obtain

$$\sin\Delta\varphi = 2\tan\tfrac{1}{2}\Delta\lambda\tan\alpha\sin\varphi' - \tan^2\tfrac{1}{2}\Delta\lambda\sin 2\varphi'\cos\Delta\varphi$$
$$- \tan^2\tfrac{1}{2}\Delta\lambda\cos 2\phi'\sin\Delta\varphi,$$

putting, for the sake of brevity, $1 + \tan^2\frac{1}{2}\Delta\lambda\cos 2\phi' = A$, $\tan^2\frac{1}{2}\Delta\lambda\sin 2\varphi' = B$, and $2\tan\frac{1}{2}\Delta\lambda\tan\alpha\sin\phi' = C$, the last equation becomes, by transposition,

$$A\sin\Delta\phi + B\cos\Delta\phi = C:$$

where A, B, C denote known quantities. By substituting for $\cos\Delta\varphi$ its value $\sqrt{(1 - \sin^2\Delta\varphi)}$, and resolving the equation, we shall determine $\sin\Delta\varphi$. Hence $\Delta\varphi$ becomes known.

We may, however, determine the value of $\Delta\varphi$ more elegantly as follows:

Dividing the above equation by A, the coefficient of its first term, it becomes,

$$\sin\Delta\varphi + \frac{B}{A}\cos\Delta\varphi = \frac{C}{A}.$$

Let θ be such an arc that $\tan\theta$, or $\frac{\sin\theta}{\cos\theta} = \frac{B}{A}$.

Then, by substituting, we have,

$$\sin \Delta\varphi + \frac{\sin\theta}{\cos\theta}\cos\Delta\varphi = \frac{C}{A};$$

$$\text{or}\quad \sin\Delta\phi\cos\theta + \cos\Delta\phi\sin\theta = \frac{C}{A}\cos\theta.$$

$$\text{that is,}\quad \sin(\Delta\phi + \theta) = \frac{C}{A}\cos\theta.$$

Since θ is an arc whose tangent is known, it is evident in what manner $\Delta\varphi$ is to be determined from this equation.

If the error in the latitude is supposed to be evanescent, we have,

$$\sin d\varphi = 2\tan\tfrac{1}{2}d\lambda\tan\alpha\sin\varphi',$$
$$\text{or}\quad d\varphi = d\lambda\tan\alpha\sin\varphi'.$$

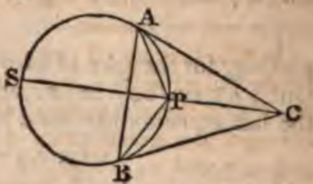
Hence it appears that if the error in the latitude be very small, the sine of the error in the time at different seasons is nearly proportional to the tangent of the sun's declination.

VII. QUESTION 397, *by* GALLICUS.

Having given the position of any number of towns whatever, in the same plane, to connect them by a system of canals, of which, the total length shall be the least possible.

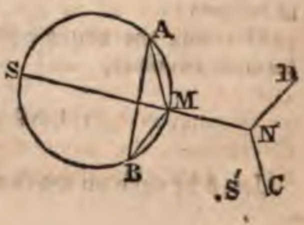
SOLUTION, *by the Proposer.*

When there are only two towns, the canal must evidently be the straight line which joins them.

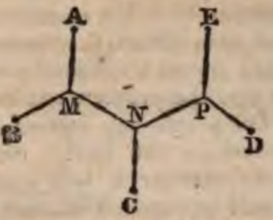


When there are three A, B, C, the canals must meet in a point P, so that the sum of the three distances AP, BP, CP, may be the least possible; which is well-known to be the case when the angles at P are equal to one another, or when each of the angles APB, APC, BPC is 120°. Simpson on the max. et min.

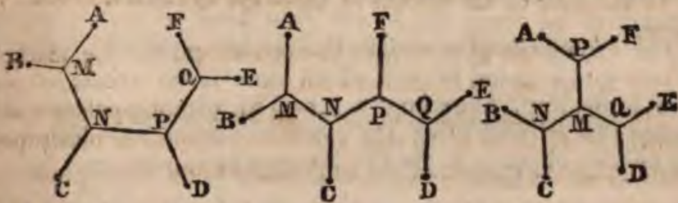
In the case of four or more towns, the canals must meet in threes, and make the angles at the points of intersection equal to one another. Thus, in the case of four towns, A, B, C, D, they must meet at M and N, and make the angles at these points equal; for the sum of the distances AM, BM, MN, CN, DN is then evidently the least possible.



In like manner for five towns, we have the following system;



and for six towns, the three systems;



in all which systems, it may be remarked, that the number of points M, N, &c. is less by two than the number of towns.

The method of construction for determining the position of the canals is in every case obvious.

First, for three points; upon the line joining A, B describe a segment of a circle to contain 120°, and bisect the circumference on the other side of AB in S, which will be a given point; then CP, when produced, will evidently pass through S, because the angles BPC, APC being equal, CP bisects the angle APB; therefore the point P where it meets the circumference of the circle will be known.

In like manner, if there are four given points, let a segment be described on AB as before, to contain 120°, and let S be the middle of the arc on the other side of AB; then MN will pass through the given point S; and if upon the line joining the points S and C, a segment be described to contain 120°, and S′ be the middle of the arc below SC, the line DN, which bisects the angle SNC, will pass through S′ and be given by position, and the points N and M will then be determined.

If there are more given points, a segment to contain 120°, must be described on the line joining S′ and the next given point, and another on the line joining S″, the middle of the last described arc, and the next given point, and so on to the last point.

We may remark, however, that when three of the lines are determined, the others may be found by drawing them respectively parallel to the first three.

Remark. When the points are more than three, the problem evidently admits of several solutions; for in the construction, the choice of the points to be taken in succession is arbitrary.

Thus in the case of four points, if we take AD or BC instead of AB or DC, we shall obtain a different position for MN, and consequently in this case the problem admits of two solutions.

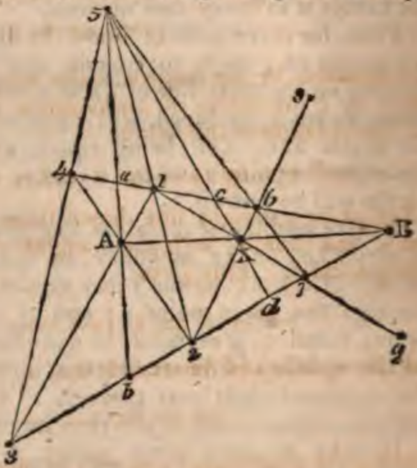
When there are five points any two of these combined will give a different solution, so that in this case there will be five solutions; and in like manner it will appear, that when there are six points there will be fourteen solutions.

VIII. QUESTION 398, *by* VICINUS.

The extremities of a straight line on the ground are marked by two upright poles, beyond which there is no access to any point in the line: Shew how a person by planting poles in the ground (but without using any other instrument) may determine any number of points in the line between its extremities.

SOLUTION, *by* A. B.

Let AB be the given straight line; place poles at any two points 1, 2, produce A1 and B2 to meet in 3, and B1 and A2 to meet in 4. Also produce 3, 4 and 2, 1 to meet in 5; take 6 any where in B1 and find 7 the point where 5, 6 produced meets B, 2. Take 8 and 9 any where in 2,6 and 1,7 produced and find x the point in which 8,2 meets 9,1 and x shall be in the straight line AB..



DEMON. Let the lines 5A and 5x be drawn to meet B,4 in *a* and *c*, and B, 3 in *b* and *d*.

It is a well-known property of a quadrilateral, that if two opposite sides be produced till they meet, the line drawn through that point and the intersection of the diagonals, meeting the other two opposite sides, is harmonically divided in the points of intersection. Therefore $5b$ is harmonically divided in the points a, A, and for the same reason $5d$ is harmonically divided in the points c, X. But, because of the harmonicals 5A, aB, AB and bB, $5d$ is also harmonically divided by the straight lines aB and AB; that is in the point C, and where it meets AB; therefore it is evident that X is in the line AB.

IX. QUESTION 399, *by* GALLICUS.

Suppose that A a point in a line of the second order is given; a point P may be found such that if any chord BC be drawn through it, the lines AB, AC shall contain a right angle.

SOLUTION, *from the Annales de Mathematiques.*

A fixed point being taken arbitrarily in a line of the second order, and if the tangent at this point be taken for the axis of x and the corresponding normal for the axis of y; and if we designate by N the length of the normal measured from its origin to the point where it again meets the curve, by

$$y = \mathrm{A}x + \mathrm{N}$$

the equation to the tangent at this extremity of the normal, and lastly by P the radius of curvature at the fixed point, we have for the equation of the curve.

$$\mathrm{N}x^2 + 2\,\mathrm{P}y\,(y - \mathrm{A}x - \mathrm{N}) = 0 \text{ .... } (1)$$

Let D be a straight line drawn from the origin of the axes x and y, and forming with them angles whose cosines are respectively a and b. This gives

$$a^2 + b^2 = 1, \text{ .... } (2)$$

and the equation of the straight line is

$$ay = bx \text{ .... } (3)$$

therefore by combining these with equation (1), we obtain the coordinates of the intersection of D with the curve

$$x = \frac{2\,\mathrm{NP}ab}{\mathrm{N}a^2 + 2\,\mathrm{P}b^2 - 2\,\mathrm{AP}ab},$$

$$y = \frac{2\,\mathrm{NP}b^2}{\mathrm{N}a^2 + 2\,\mathrm{P}b^2 - 2\,\mathrm{AP}ab}. \text{ ....} (4)$$

Again let another straight line D′, having the same origin as D, make with the axes x and y the angles whose cosines are respectively a' and b', this gives

$$a'^2 + b'^2 = 1, \quad \ldots\ldots (5)$$

and we have similarly

$$x = \frac{2\mathrm{NP}\,a'b'}{\mathrm{N}a'^2 + 2\mathrm{P}b'^2 - 2\mathrm{AP}a'b'}, \qquad \ldots\ldots (6)$$

$$y = \frac{2\mathrm{NP}b'^2}{\mathrm{N}a'^2 + 2\mathrm{P}b'^2 - 2\mathrm{AP}a'b'}.$$

From these we easily find that the equation of the chord C which joins the extremities of the two straight lines D, D′ is, when divided by $ab' - ba'$,

$$\left\{\mathrm{N}(ab' + ba') - 2\mathrm{AP}bb'\right\} x + (2\mathrm{P}bb' - \mathrm{N}aa')y = 2\mathrm{NP}bb' \quad .. (7)$$

And if for finding the points where the chord C cuts the normal and tangent, we make successively in this equation x and $y = 0$, we find

$$y = \frac{2\mathrm{NP}}{2\mathrm{P} - \mathrm{N}\dfrac{aa'}{bb'}}, \quad \ldots\ldots (8)$$

$$x = \frac{2\mathrm{NP}}{\mathrm{N}\left(\dfrac{a}{b} + \dfrac{a'}{b'}\right) - 2\mathrm{AP}} \quad \ldots\ldots (9)$$

whence it appears that when $\frac{aa'}{bb'}$ is constant, the chord C always cuts the normal in the same point, and that when $\frac{a}{a'} + \frac{b}{b'}$ is constant, the same chord always cuts the tangent in the same point, whatever may be the directions of the straight lines D and D′.

Among the divers cases in which $\frac{aa'}{bb'}$ is constant, the most simple is, without doubt, that where we have

$$aa' + bb' = 0, \text{ or where } \frac{aa'}{bb'} = -1;$$

the straight lines D, D′ are then perpendicular to each other, and the fixed point in the normal through which the straight line C passes is given by the formula

$$y = \frac{2\mathrm{NP}}{2\mathrm{P} + \mathrm{N}}.$$

Whence there results the theorem in the question.

X. QUESTION 400, *by* PEREGRINATOR.

No answer to this question has been received.

XI. QUESTION 401, *by Mr.* W. WALLACE, R. M. C.

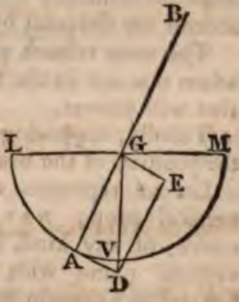
A beam is supported by resting on a prop, and with one end on a curve surface: Find the nature of the curve so that the beam may be in equlibrio in any position whatever. Also supposing the nature of the curve given, find the position of the beam when it is in equlibrio.

SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Either a circle or a conchoid, will satisfy the first part of the problem.

In the first place, we shall shew that a circle will satisfy the first part of the problem.

Let AB represent the beam, the point G being its centre of gravity: with the centre G and radius GA describe a semicircle LVM, in a vertical plane, the diameter LM being parallel to the horizon, and the radius GV at right angles thereto; and let VG represent the prop. Then the beam being placed with its end A upon the circumference of the circle, and leaning with its centre of gravity G, against the top of the prop, will rest in any position.



Draw AD perpendicular to AG, meeting GV in D, and complete the rectangle AGED.

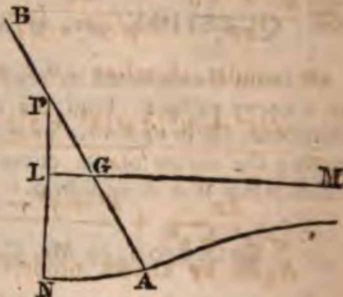
When the beam is in equilibrio, by the principles of statics, the weight thereof, its pressure against the circle at A, and its pressure against the top of the prop at G, will be as GD, GA $=$ ED, and GE $=$ AD respectively, and in the several directions of those lines.

Now the pressure in the direction GA, is sustained by the reaction of the circle, in the contrary direction AG: and the pressure in the direction GE, is sustained by the reaction of the prop VG, in the contrary direction; therefore, by the well-known principles of statics, the beam will be in equilibrio.

When G, is not in the middle of AB there will obviously be two circles that will answer; viz. either the circle described with the radius GA, or that described with the radius GB.

In the next place we shall shew that a conchoid will satisfy the first part of the problem.

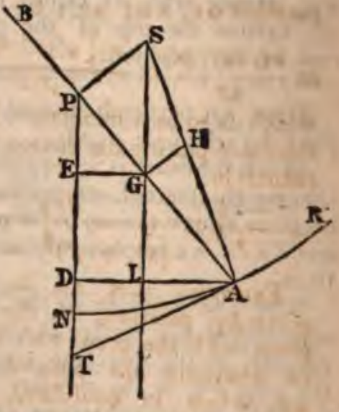
Let LP be the given prop, perpendicular to the horizon, and LM a right line at right-angles thereto, or parallel to the horizon. Imagine the beam AB, to be placed leaning against the prop at P, and having its centre of gravity G, always in the line LM; the locus of the end A, will be the curve, which is called the superior conchoid, as is manifest from the generation of that curve, and therefore the beam AB being placed with its end A, upon the curve, and leaning against the top of the prop at P, will, in every position thereof, have its centre of gravity in the line LM: and the beam being so placed can have no tendency to move, as the centre of gravity will neither ascend nor descend by altering its inclination.



The same remark applies here as in the case of the circle; viz. when G is not in the middle of AB, there will be two conchoids that will answer.

Thirdly, supposing the nature of the curve to be given; find the position of the beam when in equilibrio.

Let AB represent the beam, G its centre of gravity; PDT a vertical prop; NAR a given curve, of any kind, in the same vertical plane with the prop. Let AT be a tangent to the curve at A; AS perpendicular to AT, and PS perpendicular to AB; also draw GH parallel to PS meeting AS in H. When the beam is in equilibrio SG will be perpendicular to the horizon; and the weight of the beam, its pressure against the curve at A, and its pressure against the top of the prop PDT, will be as the lines SG, SH and HG respectively, and in the several directions of those lines. See Emerson's smaller book of mechanics, prop. 37 and cor.



Let AD be perpendicular to PT, and intersect SG in L; put $AG = d$, $PD = x$, $DA = y$; then $AP = \sqrt{(x^2 + y^2)}$ and $PG = AP - AG = \sqrt{(x^2 + y^2)} - d$.

Because of the parallels PD, GL; AP : AG :: AD : AL $= \frac{AG \times AD}{AP} = \frac{dy}{\sqrt{(x^2+y^2)}}$; also AP : AG :: PD : GL $= \frac{AG \times PD}{AP} = \frac{dx}{\sqrt{(x^2+y^2)}}$. The triangles ADP, and PGS are similar, therefore PD : AP :: PG : GS $= \frac{AP \times PG}{PD}$ $= \frac{\sqrt{(x^2+y^2)}\left\{\sqrt{(x^2+y^2)} - d\right\}}{x}$, and hence LS $=$ GL $+$ GS

$$= \frac{dx}{\sqrt{(x^2+y^2)}} + \frac{x^2+y^2-d\sqrt{(x^2+y^2)}}{x} = \frac{(x^2+y^2)^{\frac{3}{2}} - dy^2)}{x\sqrt{(x^2+y^2)}}.$$

Again by known properties of tangents, demonstrated by writers on fluxions $-\dot{x} : \dot{y} ::$ AL $= \frac{dy}{\sqrt{(x^2+y^2)}}$: LS $= \frac{(x^2+y^2)^{\frac{3}{2}} - dy^2}{x\sqrt{(x^2+y^2)}}$, whence $\frac{-\dot{x}}{\dot{y}} = \frac{dxy}{(x^2+y^2)^{\frac{3}{2}} - dy^2}$.

From what has been here deduced, and the given equation of the curve, the values of x and y may be found, and the required position of AB thereby determined.

The same conclusion will be derived from the known statical principle, that when the equilibrium takes place, the centre of gravity will be at its highest or lowest point, with respect to an horizontal line given by position.

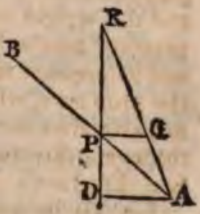
Draw GE parallel to AD meeting PT in E: because of the parallels GE, AD; AP : GP :: PD : PE

$$= \frac{PD \times GP}{AP} = \frac{x\left\{\sqrt{(x^2+y^2)} - d\right\}}{\sqrt{(x^2+y^2)}} = x - \frac{dx}{\sqrt{(x^2+y^2)}}.$$

which is to be a maximum, by the principle above alluded to; therefore putting the fluxion of the expression $= 0$, after proper reduction

$$\frac{-\dot{x}}{\dot{y}} = \frac{dxy}{(x^2+y^2)^{\frac{3}{2}} - dy^2},$$ the very same as before.

Example. Let AR be a right line given by position, in the same vertical plane with the prop PD: It is required to determine the position of the beam AB, when in equilibrio, resting upon the prop at P, and having one end A, upon the line AR.



Let DP meet AR in R; draw PQ parallel to DA meeting AR in Q: put PR $= a$, PQ $= b$, PD $= x$ and DA $= y$; then RD $= a + x$. Because of the parallels PQ, DA; RP : PQ :: RD : DA, whence RP × DA = PQ × RD, that is, $ay = b\,(a + x)$, $\therefore a\dot{y} = b\,\dot{x}$, $\frac{-\dot{x}}{\dot{y}} = \frac{-a}{b}$; and hence the expression $\frac{-\dot{x}}{\dot{y}}$ $= \frac{dxy}{(x^2 + y^2)^{\frac{3}{2}} - dy^2}$, becomes $\frac{dxy}{(x^2 + y^2)^{\frac{3}{2}} - dy^2} = \frac{-a}{b}$, or $bdxy = ady^2 - a\,(x^2 + y^2)^{\frac{3}{2}}$, from which equation and $ay = b\,(a + x)$, x and y may be found, and the required position of the beam determined.

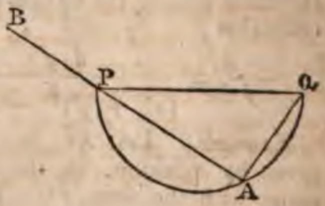
In some cases it will be convenient to express the above deduced property in the following manner:

Put AG $= d$, AP $= v$, then PG $= v - d$; also put sin ∠ PAD (PGE) $= s$, its cosine $= c$; radius 1.

By trigonometry PE $= s \times (v - d)$, which is to be a maximum by the question; therefore putting its fluxion $= 0$, $\dot{s} \times (v - d) + s\dot{v} = 0$, or $s\dot{v} = -\dot{s} \times (v - d)$. Now $s^2 + c^2 = 1$, and, taking the fluxions, $s\dot{s} + c\dot{c} = 0$, whence $-\dot{s}$ $= \frac{c\dot{c}}{s}$, and by means of this, the expression, $s\dot{v} = -\dot{s} \times (v - d)$ becomes $s\dot{v} = \frac{c\dot{c}}{s}(v - d)$, or $s^2\dot{v} = c\dot{c}\,(v - d)$; from which property and the nature or equation of the given curve, the position of the beam AB, when in equlibrio, may be determined.

Example. Suppose the given curve to be a semicircle, and the top of the prop or given point P, to be at the end of an horizontal diameter PQ.

Join QA and put PQ $= 2r$; then, by trigonometry, PA $= v$ $= 2rc$, $\dot{v} = 2r\dot{c}$; and by means of these, the expression $s^2\dot{v} =$ $c\dot{c}\,(v - d)$ becomes $2rs^2\dot{c} = c\dot{c}$ $(2rc - d)$, $2rs^2 = c\,(2rc - d)$ $= 2rc^2 - dc$, $dc = 2r\,(c^2 - s^2)$ $= 2r\,(2c^2 - 1)$ from which quadratic equation c may be found, and the position of the beam be thereby determined.

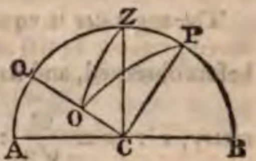


XII. QUESTION 402, *by Mr.* CUNLIFFE, *R. M. C.*

At the time of the equinox, in latitude 60°, suppose a person sets out at noon, on a clear day, and travels till sun set, in the direction of his shadow, at the uniform rate of five miles an hour; how far will he then be from the place from whence he set out, supposing the journey performed upon an horizontal plane?

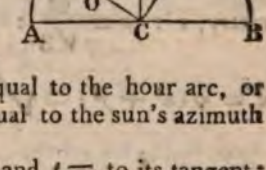
SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Previous to the solution of the question, it will be proper to put down the way of finding the relation between the hour angle, and the azimuth, for the given latitude and place of the sun.



Let the semicircle AQZPB represent the meridian of the place, or hour line of 12 o'clock; the diameter AB the horizon; P the pole; Z the zenith; and Q a point in the equator; and draw CP, CQ. Let O, in CQ, represent the place of the sun, and imagine the great circles PO and ZO to be drawn. In the right-angled spherical triangle OQZ, the leg QZ is equal to the latitude of the place; the leg OQ is equal to the hour arc, or angle from 12; and the angle QZO is equal to the sun's azimuth from the south

Put z = OQ, the hour angle from 12, and $t =$ to its tangent: also put m = tan. of ∠ QZO, the sun's azimuth, radius 1. By spherical trigonometry, tan QO = sin QZ × tan QZO, and hence tan QZO $= \frac{\tan \text{QO}}{\sin\ \text{QZ}} = \frac{2t}{\sqrt{3}}$; and, by the principles of fluxions, $\dot{z} = \frac{\dot{t}}{1 + t^2}$. Having found the relation of the azimuth and hour angle, we shall now proceed to the solution of the question.



Let the curve ACc represent a portion of the traveller's path; AD a right line drawn due east, from the point A, from whence he set out. Let DC, dc be two rectangular co-ordinates to the curve indefinitely near to each other, and Ce parallel to AD. Put AD $= x$, DC $= y$, and the curve AC $= u$; then Dd = Ce $= \dot{x}$, $ec = \dot{y}$, and Cc $= \dot{u}$.

By the question, the traveller proceeds in the direction of his shadow Cc; wherefore the ∠ Cce is equal to the sun's azimuth from the south, and its tangent $m = \frac{2t}{\sqrt{3}}$, as before found.

Now as the traveller moves uniformly in his course at the given rate of 5 miles an hour, the distance AC will obviously have a given ratio to the time, or hour arc from noon: therefore put $u = ax$, a being a given quantity to be determined from the rate of walking. The hour arc, from noon till sun set, at the time of the equinox is a quadrant, which denote by q; and the distance travelled in that time will be 30 miles; therefore $aq = 30$, whence $a = 30 \div q = 19{\cdot}0985485$. By taking the fluxion of the assumed expression $u = az$, we get

$$\dot{u} = a\dot{z} = \frac{a\dot{t}}{1 + t^2}.$$

The angle Cce is equal to the sun's azimuth from the south, as before observed, and its tangent $m = \frac{2t}{\sqrt{3}}$, therefore, by trigonometry, $1 : m = \frac{2t}{\sqrt{3}} :: \dot{y} : \dot{x}$, whence $\dot{x} = \frac{2t\dot{y}}{\sqrt{3}}$; and hence $\dot{u} = \sqrt{(\dot{x}^2 + \dot{y}^2)} = \sqrt{\left(\frac{4t^2\,\dot{y}^2}{3} + \dot{y}^2\right)} = \frac{\dot{y}}{\sqrt{3}}\sqrt{(4t^2 + 3)}$; and therefore $\frac{\dot{y}}{\sqrt{3}}\sqrt{(4t^2 + 3)} = \frac{a\dot{t}}{1+t^2}$, whence $\dot{y} = \frac{a\sqrt{3} \times \dot{t}}{(1+t^2)\sqrt{(4t^2+3)}}$, and hence $\dot{x} = \frac{2t\dot{y}}{\sqrt{3}} = \frac{2at\dot{t}}{(1 + t^2)\sqrt{(4t^2 + 3)}}$.

First, to find the fluent of $\dot{x} = \frac{2at\dot{t}}{(1 + t^2)\sqrt{(4t^2 + 3)}}$.

Put $4t^2 + 3 = v^2$, $t^2 = \frac{v^2 - 3}{4}$, $1 + t^2 = 1 + \frac{v^2 - 3}{4} = \frac{1 + v^2}{4}$, $4t\dot{t} = v\dot{v}$, and hence $\dot{x} = \frac{2at\dot{t}}{(1 + t^2)\sqrt{(4t^2 + 3)}} = \frac{2a\dot{v}}{1 + v^2}$, and, taking the fluents, $x = 2a \times$ circular arc radius 1 and tan $v = \sqrt{(4t^2 = 3)}$; but when $t = 0$, $x = 0$, and therefore the correct equation of the fluents will be $x = 2a \times$ A, where A denotes the circular arc, radius 1, and tangent $\frac{\sqrt{(4t^2 + 3)} - \sqrt{3}}{1 + \sqrt{3} \times \sqrt{(4t^2 + 3)}}$. Now at sun set, t becomes infinite, and then A becomes a circular arc, radius 1 and tangent $\frac{1}{\sqrt{3}}$, or $\frac{1}{3}$ of a quadrant q, at which time the expression for x, or the greatest abscissa AD, becomes $\frac{2}{3}aq = 20$ miles.

We shall now shew a method of finding the fluent of the expression $\dot{y} = \frac{a\sqrt{3} \times \dot{t}}{(1 + t^2)\sqrt{(4t^2 + 3)}}$.

Put $4t^2 + 3 = (s - 2t)^2 = s^2 - 4st + 4t^2$. whence $t = \frac{s^2 - 3}{4s}$. $\sqrt{(4t^2 + 3)} = s - 2t = s - \frac{s^2 - 3}{2s} = \frac{s^2 + 3}{2s}$, and by taking the fluxion of the expression $t = \frac{s^2 - 3}{4s}$, $\dot{t} = \frac{\dot{s} \times (s^2 + 3)}{4s^2}$; $1 + t^2 = 1 + \frac{(s^2 - 3)^2}{16s^2} = \frac{s^4 + 10s^2 + 9}{16s^2}$, $\frac{\dot{t}}{1 + t^2} = \frac{4\dot{s} \times (s^2 + 3)}{s^4 + 10s^2 + 9}$; and hence

$$\dot{y} = \frac{a\sqrt{3} \times \dot{t}}{(1 + t^2)\sqrt{(4t^2 + 3)}} = \frac{a\sqrt{3} \times 8\dot{s}s}{s^4 + 10s^2 + 9};$$

and taking the fluents

$$y = \frac{a\sqrt{3}}{2} \times h.l.\ \frac{s^2 + 1}{s^2 + 9} = \frac{a\sqrt{3}}{2} \times h.l.\ \frac{2t^2 + 1 + t\sqrt{(4t^2 + 3)}}{2t^2 + 3 + t\sqrt{(4t^2 + 3)}}.$$

But when $t = 0$, $y = 0$, and therefore the correct equation of the fluents will be

$$y = \frac{a\sqrt{3}}{2} \times h.l.\ \frac{2t^2 + 1 + t\sqrt{(4t^2 + 3)}}{2t^2 + 3 + t\sqrt{(4t^2 + 3)}} + \frac{a\sqrt{3}}{2} \times h.l.\ 3.$$

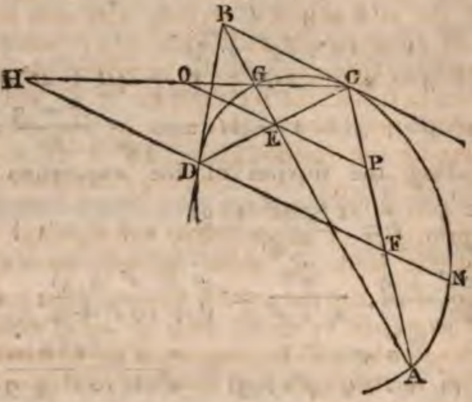
Now at sun set, t is infinite, being the tangent of 90°, or a quadrant, in which circumstance the expression for the ordinate DC, becomes barely $\frac{a\sqrt{3}}{2} \times h.l.\ 3 = 18{\cdot}17085859$ miles; and from the two legs 20 and 18 17085859 of a right-angled triangle we find the hypothenuse $= 27{\cdot}02218449$ miles, being the traveller's distance at sun set from the place from whence he started at noon.

XIII. QUESTION 403, *by* AMICUS.

If, from a point without a conic section, two tangents be drawn to the curve, and also any other straight line to meet it in two points, the lines joining these points and one of the points of contact shall cut off equal segments from the chord which passes through the other point of contact parallel to the tangent.

SOLUTION, *by* AMICUS, *the Proposer.*

Let BC and BD be tangents to a conic section, and let any line be drawn through B, to meet the section in G and A; also let CG and CA be drawn to meet the chord DN, which is parallel to BC, in F and H; the segment DH is = DF.



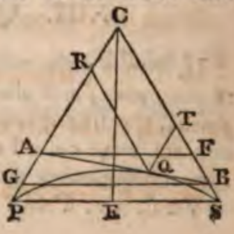
Join DC intersecting BA in E, and through E draw OEP parallel to HN or BC meeting HC in O and CA in P. Because OP is parallel to BC, we have BC : OE :: BG : GE, and BC : EP :: BA : EA; but, by the property of tangents, BG : BA :: GE : EA, and alternately BG : GE :: BA : EA; therefore, by equality of ratios, BC : OE :: BC : EP; wherefore OE is = EP and consequently DH is = DF.

XIV. QUESTION 404, *by Mr.* CUNLIFFE, *R. M. C.*

In a given hollow cone, it is required to place a solid, the axis of which shall concide with the axis of the cone, and the property of the solid shall be such, that the area of any ellipse, formed by a plane touching the solid and cutting the cone, shall be of the same given magnitude; required the equation of the curve by whose revolution the solid will be generated.

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let the figure CPQS, represent a section of the cone and solid, by a plane passing through their common axis CE: and let AB be a tangent to the curve PS, at the point Q, meeting CP, CS, in A and B respectively. Draw QT parallel to CA, and QR parallel to CB; also draw AF and BG at right angles to the axis CE.



It is manifest that AB, will be the transverse axis of the elliptical section formed by a plane cutting the cone and touching the solid, in the point Q.

And it is well-known that the rectangle BG × AF, is equal to the square of the conjugate axis of that section; wherefore it is obvious that $\text{AB} \surd (\text{BG} \times \text{AF})$ will have a given ratio to the area of the said elliptical section, and will therefore be given. And consequently $\text{AB}^2 \times \text{BG} \times \text{AF}$, being the square of the preceding expression will also be given: Moreover as the cone is given in species the rectangle BG × AF will manifestly have a given ratio to the rectangle CB × CA; wherefore, $\text{AB}^2 \times \text{BG} \times \text{AF}$, being by the question given, $\text{AB}^2 \times \text{CB} \times \text{CA}$ will be given also. Put $\text{CB} = v$, $\text{CA} = u$, $\text{CR} = \text{QT} = x$, and $\text{CT} = \text{RQ} = y$; also let $c = \cos. \angle \text{ACB}$, radius 1.

By the principles of trigonometry and fluxions, the angular fluxion, of the point B, about Q, will be expressed by $\sin \angle \text{CBQ} \times \dot{v}$: also the angular fluxion of the point A, about Q, will be expressed by $\sin \angle \text{CAQ} \times -\dot{u}$, radius being 1. The latter expression will be negative, if the former be positive; for whilst CB increases, CA will manifestly decrease.

By obvious principles

$$\text{AQ} : \text{BQ} :: \text{AR} : \text{CR} :: \sin \text{CAQ} \times -\dot{u} : \sin \text{CBQ} \times \dot{v}.$$

And by trigonometry

$$\text{CB} = v : \text{CA} = u :: \sin \text{CAQ} : \sin \text{CBQ};$$

and hence the preceding analogy becomes

$$\text{AR} : \text{CR} :: -v\dot{u} : u\dot{v}; \text{ and, by composition,}$$

$$\text{CA} = u : \text{CR} = x :: u\dot{v} - v\dot{u} : u\dot{v}$$

$$\text{whence} \quad x = \frac{u^2\dot{v}}{u\dot{v} - v\dot{u}} = \frac{u^2}{u - \frac{v\dot{u}}{\dot{v}}}.$$

Again $\text{CA} : \text{CR} :: \text{CB} = v : \text{BT} = v - y :: u\dot{v} - v\dot{u} : u\dot{v}$,

whence
$$y = \frac{v^2}{v - \frac{u\dot{v}}{\dot{u}}}.$$

From the two preceding equations, we readily get

$\frac{x}{y} = \frac{-u^2\dot{v}}{v^2\dot{u}}$, whence $-\dot{u} = \frac{yu^2\dot{v}}{xv^2}$; and this being written for $-\dot{u}$, in the first equation, gives

$$xv + yu = vu.$$

By a well-known property

$\text{AB}^2 = \text{CB}^2 - 2\text{CB} \times \text{AC} \times c + \text{AC}^2 = v^2 - 2vuc + u^2$; and hence the expression

$\text{CB} \times \text{CA} \times \text{AB}^2 = vu \times (v^2 - 2vuc + u^2)$ which has been shewn to be given; wherefore, put $uv \times (v^2 - 2vuc + u^2) = a^4$. By putting the preceding expression into fluxions, after proper reduction, we shall get

$$-\frac{\dot{u}}{\dot{v}} = \frac{u \times (v^2 - 2vuc + u^2) + 2v^2u - 2vu^2c}{v \times (v^2 - 2vuc + u^2) + 2vu^2 - 2v^2uc};$$ and hence

$$x = \frac{u^2}{u - \frac{v\dot{u}}{\dot{v}}} = \frac{u^2}{u + \frac{u \times (v - 2vuc + u^2 + 2v^2u - 2vu^2c}{v^2 - 4vuc + 3u^2}}$$

$$= \frac{u \times (v^2 - 4vuc + 3u^2)}{4 \times (v^2 - 2vuc + u)}.$$ Also $y = \dfrac{v^2}{v - \frac{u\dot{v}}{\dot{u}}}$

$$= \frac{v^2}{v + \frac{v \times (v^2 - 2vuc + u^2) + 2vu^2 - 2v^2uc}{3v^2 - 4vuc + u^2}} = \frac{v \times (3v^2 - 4vuc + u^2)}{4 \times (v^2 - 2vuc + u^2)}.$$

Again, from the equation $vu = yu + xv$ before deduced we get, $u = \frac{xv}{v - y}$, and this being written for u in the expression

$$x = \frac{u \times (v^2 - 4vuc + 3u^2)}{4 \times (v^2 - 2vuc + u^2)},$$ it becomes

$$x = \frac{xv}{v - y} \times \frac{v^2 - \frac{4cv^2x}{v - y} = \frac{3v^2x^2}{(v - y)^2}}{4 \times \left\{v^2 - \frac{2cv^2v}{v - y} + \frac{v^2x}{(v - y)^2}\right\}};$$ whence

$$4(v-y)\left\{(v-y)^2 - 2cx(v-y) + x^2\right\} = v\left\{(v-y)^2 - 4cx(v-y) + 3x^2\right\}:$$

and hence

$$(3v-4y) \times (v-y)^2 - 8cx \times (v-y)^2 + 4cvx \times (v - y + vx^2 - 4x^2y = 0,$$

which being reduced and arranged according to the powers of v, gives

$$3v^3 - 2v^2(5y + 2cx) + v(x^2 + 12cxy + 11y^2) - 4y(x^2 + 2cxy + 2y^2) = 0,$$

from which equation v must be found in terms of x and y, and having found v in terms of x and y, we shall get u in terms of x and y from the equation $u = \frac{vx}{v - y}$; and writing the said values of v and u, in the equation $vu \times (v^2 - 2vuc + u^2) = a^4$, we shall have the equation of the curve required in terms of its own co-ordinates.

Solutions to a number of curious problems may be easily obtained from what has been done.

We shall subjoin a few.

1. Let it be required to find the equation of the curve which shall touch the base of a plane triangle whose vertical angle is formed by two lines given by position, and the sum of the including sides a given length.

Put $v + u = a$, the given length, whence $\dot{v} + \dot{u} = 0$, $\frac{-\dot{u}}{\dot{v}} = 1$; and hence $x = \frac{u^2}{u - \frac{v\dot{u}}{\dot{v}}} = \frac{u^2}{u + v} = \frac{u^2}{a}$, therefore

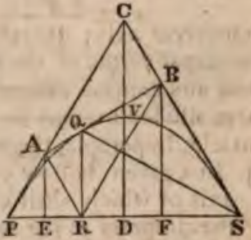
$u = a^{\frac{1}{2}} x^{\frac{1}{2}}$. Also $y = \frac{v^2}{v - \frac{u\dot{v}}{\dot{u}}} = \frac{v^2}{v + u} = \frac{v^2}{a}$, therefore

$$v = a^{\frac{1}{2}} y^{\frac{1}{2}} \text{ and hence } v + u = a^{\frac{1}{2}} y^{\frac{1}{2}} + a^{\frac{1}{2}} x^{\frac{1}{2}} = a;$$

dividing by $a^{\frac{1}{2}}$, $y^{\frac{1}{2}} + x^{\frac{1}{2}} = a^{\frac{1}{2}}$, which equation belongs to the common conic parabola.

The foregoing property of the parabola, may be verified in the following manner:

Let PCS be an isosceles triangle, and PQS a parabola touching the equal sides CP, CS, in the points P and S; also let AB, be another line touching the curve in Q, and terminating in the lines CP, CS in A and B, respectively: then AC + CB = CP.



Demonstration. Draw AE, QR, CD and BF perpendicular to PS; and join AR, BR.

By a known property of the curve, PE = ER, and BF = FS; and therefore AR = AP; and AR is parallel to BC. In like manner BR = BS; and BR is parallel to AC. Wherefore the figure CARB is a parallelogram, and therefore CB = AR = AP; and consequently CB + CA = AP + CA = CP.

2. In a given hollow cone, it is required to place a solid, the axis of which shall coincide with the axis of the cone, and the property be such, that the conjugate axis of every ellipse, formed by a plane touching the solid and cutting the cone, shall be of the same given magnitude; required the equation

the equation of the curve by whose revolution the solid will be generated.

By attending to what has been deduced we shall perceive that the square of the conjugate axis of the elliptical section will be proportional to CB × CA $= vu$; therefore put $vu = a^2$ a given quantity; taking the fluxions $v\dot{u} + u\dot{v} = 0$,

whence $\frac{-\dot{u}}{\dot{v}} = \frac{u}{v}$, and $x = \frac{u^2}{u - \frac{v\dot{u}}{\dot{v}}} = \frac{u^2}{2u} = \frac{u}{2}$, and hence $u = 2x$. Also $y = \frac{v^2}{v - \frac{u\dot{v}}{\dot{u}}} = \frac{v^2}{2v} = \frac{v}{2}$, and hence $v = 2y$, and therefore $vu = 4xy = a^2$, which equation expresses the property of a conic hyperbola with respect to the asymptotes CF, and CA; and consequently an hyperbolic conoid is the required solid,

The particular hyperbola, and its position with respect to the asymptotes, will be determined as follows: we have found $4xy = a^2$, $xy = \frac{a^2}{4}$: now suppose x and y equal to each other, then $x^2 = \frac{a^2}{4}$, and $x = \frac{a}{2}$, which is equal to the semi-transverse axis; therefore we have given the asymptotes and principal vertex of the hyperbola, to describe the same. A very neat and concise demonstration of the property of the hyperbola here alluded to, may be found, in Dr. Hutton's course, vol. III. article, hyperbola, theorem 28.

3. In a given hollow cone, it is required to place a solid, the axis of which shall concide with the axis of the cone, and the property of the solid be such, that the content of the part of the cone, intercepted between its vertex and a plane touching the solid in any point, shall be of the same given magnitude; required the equation of the curve by whose revolution the solid will be generated.

From what has been done we shall perceive that the area of the elliptical section of the cone formed by a plane touching the solid will be proportional to AB $\sqrt{(\text{CB} \times \text{CA})}$, and the perpendicular from C upon AB will be proportional to $\frac{\text{CB} \times \text{CA}}{\text{AB}}$, the angle ACB, being given. And therefore, the content of the part of the cone intercepted between its vertex and a plane touching the solid, will be proportional to CB × CA × $\sqrt{(CB \times CA)} = vu\sqrt{vu}$, which is to be a given quantity by the question. Put $vu\sqrt{vu} = a^3$, whence $vu = a^2$, the very same as in the preceding problem.

XV. QUESTION 405, *by Mr.* W. WALLACE, *R. M. C.*

Let a be an arc of a circle, and c its chord; also let c', c'', c''', c^{iv}, &c, be the chords of its half, its fourth, its eight, its sixteenth, &c. Prove that

$$\sin a = c - c\,c'\,c'' + c\,c'\,c''\,c'''\,c^{\text{iv}} - c\,c'\,c''\,c'''\,c^{\text{iv}}\,c^{\text{v}}\,c^{\text{vi}} + \&c.$$

$$\cos a = 1 - c\,c' + c\,c'\,c''\,c''' - c\,c'\,c''\,c'''\,c^{\text{iv}}\,c^{\text{v}} + \&c.$$

both series being continued *ad infinitum*.

SOLUTION *by Mr.* W. WALLACE, *the Proposer.*

By the arithmetic of sines

$$\begin{array}{ll}
\sin a = c \cos \tfrac{1}{2} a & \cos a = 1 - c \sin \tfrac{1}{2} a \\
\sin \tfrac{1}{2} a = c' \cos \tfrac{1}{4} a & \cos \tfrac{1}{2} a = 1 - c' \sin \tfrac{1}{4} a \\
\sin \tfrac{1}{4} a = c'' \cos \tfrac{1}{8} a & \cos \tfrac{1}{4} a = 1 - c'' \sin \tfrac{1}{8} a \\
\sin \tfrac{1}{8} a = c''' \cos \tfrac{1}{16} a & \cos \tfrac{1}{8} a = 1 - c''' \sin \tfrac{1}{16} a \\
\sin \tfrac{1}{16} a = c^{\text{iv}} \cos \tfrac{1}{32} a & \cos \tfrac{1}{16} a = 1 - c^{\text{iv}} \sin \tfrac{1}{32} a \\
\quad\&c. & \quad\&c.
\end{array}$$

Hence by continual substitution,

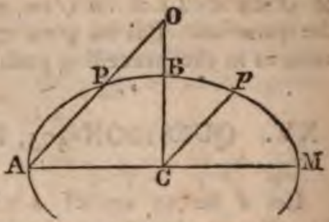
$$\begin{aligned}
\sin a &= c - c\,c' \sin \tfrac{1}{4} a \\
&= c - c\,c'\,c'' \cos \tfrac{1}{8} a \\
&= c - c\,c'\,c'' + c\,c'\,c''\,c''' \sin \tfrac{1}{16} a \\
&= c - c\,c'\,c'' + c\,c'\,c''\,c'''\,c^{\text{iv}} \cos \tfrac{1}{32} a
\end{aligned}$$

and so on continually. By a like manner of proceeding,

$$\begin{aligned}
\cos a &= 1 - c\,c' \cos \tfrac{1}{4} a \\
&= 1 - c\,c' + c\,c'\,c'' \sin \tfrac{1}{8} a \\
&= 1 - c\,c' + c\,c'\,c''\,c''' \cos \tfrac{1}{16} a \\
&= 1 - c\,c' + c\,c'\,c''\,c''' - c\,c'\,c''\,c'''\,c^{\text{iv}} \sin \tfrac{1}{32} a.
\end{aligned}$$

The two series of formulae exhibit an infinite number of elegant expressions for the sine and cosine of an arc, and, ultimately, they give the infinite series proposed in the question.

XVI. QUESTION 406, *by Mr.* HERSCHEL.



AB, CB are the major and minor semi-axes of a conic section, C*p* any semi-diameter. Draw APO parallel to C*p* meeting CB, produced if necessary, in O. It is required to prove that OA. AP $= 2\text{C}p^2$.

SOLUTION, *by Mr.* HERSCHEL, *the Proposer.*

Let $\text{C}p = r$, $\text{AP} = \text{R}$, $\angle p\text{CM} = \text{OAC} = \theta$, $\text{AC} = a$, $\text{CB} = a\sqrt{1-e^2}$, then the two polar equations of the conic section about C and A are respectively

$$r^2 = \frac{a^2(1-e^2)}{1-e^2.\cos\theta},$$

$$\text{R} = \frac{2a(1-e^2).\ \cos\theta}{1-e^2.\cos\theta}.$$

These expressions are easily deducible from the well-known equations $(1-e^2)(a^2-x^2) = y^2$ and $(1-e^2)(2ax-x^2) = y^2$ by the usual rules for transfering rectangular into polar co-ordinates. Hence we have

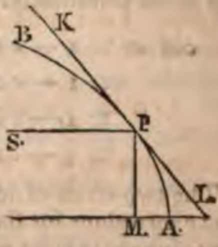
$$2r^2 = \text{R}\ \frac{a}{\cos\ \theta}.$$

Now since the angle OAC $= \theta$ and AC $= a$, therefore OA $= \frac{a}{\cos\theta}$; whence, $2\text{C}p^2 = \text{OA}\ .\ \text{AP}$.

XVII. QUESTION 407, *by Mr.* HERSCHEL.

The wind blows a spider's web hanging freely upon two points (and supposed destitute of relative gravity) into a curvilinear form. Required the nature of the curve and the law of tension.

SOLUTION, *by Mr.* HERSCHEL, *the Proposer.*



Take A the vertex, let $\text{AM} = x$, $\text{MP} = y$, $\text{AP} = s$. and $t =$ tension at P. Then the effect of the wind on a particle ds situated at P is resolved into $ds \times (\text{sin inclin})^3 = ds.\left(\frac{dy}{ds}\right)^3$ in the direction SP and $ds \times (\text{sin inclin})^2 \times \cos\ (\text{inclin}) = ds.\frac{dy^2\,dx}{ds^3}$ in the direction MP. The particle ds then is kept at rest by four forces, viz.; t in the direction of the tangent PL, $t + dt$ in that of the consecutive tangent PK and the two forces above enumerated. Let the tangential forces be reduced to the directions AM and PM, and they become respectively

$$-t.\frac{dx}{ds} \text{ and } + t.\frac{dy}{ds}$$

$$+\left\{t\frac{dx}{ds} + d\left(t\frac{dx}{ds}\right)\right\} \text{ and } -\left\{t\frac{dy}{ds} + d\left(t\frac{dy}{ds}\right)\right\}$$

the other forces estimated in the same directions are

$$-ds.\left(\frac{dy}{ds}\right)^3 \text{ and } - ds.\frac{dy^2\,dx}{ds^3}.$$

Hence the conditions of equilibrium are

$$0 = \left\{t\frac{dx}{ds} + d\left(t\frac{dx}{ds}\right)\right\} - t\frac{dx}{ds} - ds\left(\frac{dy}{ds}\right)^3$$

$$0 = -\left\{t\frac{dy}{ds} + d\left(t\frac{dy}{ds}\right)\right\} + t\frac{dy}{ds} - ds.\frac{dy^2\,dx}{ds^3};$$

that is,

$$0 = \frac{dy^3}{ds^2} - d\left(t\frac{dx}{ds}\right); \qquad (1)$$

$$0 = \frac{dy^2\,dx}{ds^2} + d\left(t\frac{dy}{ds}\right); \qquad (2)$$

Let (1) be multiplied by dx and (2) by dy, and the difference of the results will be

$$0 = dt.\frac{dx^2 + dy^2}{ds^2} + t.\frac{dx\,d^2x + dy\,d^2y}{ds^2}$$

d^2s being supposed zero; that is, since $dx^2 + dy^2 = ds^2$, and $dxd^2x + dyd^2y = 0$,

$$dt = 0, \quad t = a.$$

Again. Let (1) be multiplied by dy and (2) by dx, and their sum will be

$$0 = \frac{dy^2}{ds^2}(dy^2 + dx^2) + a\,(dxd^2y - dyd^2x)$$

and eliminating d^2x by the equation $dxd^2x + dyd^2y = 0$,

$$0 = \left(\frac{dy}{ds}\right)^2 + a.\frac{d^2y}{dxds}.$$

Suppose $\frac{dy}{ds} = p$, then $\frac{dx}{ds} = \sqrt{1 - p^2}$ and

$$0 = p^2 + a \cdot \frac{dp}{ds \cdot \sqrt{1-p^2}}$$

which gives $$ds = -a \cdot \int \frac{dp}{p^2\sqrt{1-p^2}}.$$

Hence, $x = \int ds \cdot \sqrt{1-p^2} = \int -\frac{adp}{p^2}$, or $x + c = \frac{a}{p}$

$$c' + y = \int pds = \int -\frac{adp}{p\sqrt{1-p^2}}$$

or, $$y + c' = a.\log\left\{\frac{1}{p} + \sqrt{\frac{1}{p^2} - 1}\right\}$$

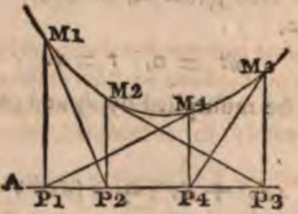
and reducing (supposing, for simplicity, x and y put for $x + c$, and $y + c'$ which does not diminish the generality of the equation),

$$\frac{x}{a} = \frac{e^{\frac{y}{a}} + e^{-\frac{y}{a}}}{2}$$

which, by the solution of question 365 of the last No. of the Repository, appears to belong to the catenary. The tension is the same in every point.

XVIII. QUESTION 408, *by Mr.* HERSCHEL.

To determine the nature of a curve such, that drawing any ordinate $P_1 M_1$ and tangent $M_1 P_2$, again erecting a second ordinate $P_2 M_2$ and drawing a second tangent $M_2 P_3$, if this be repeated n times, the last tangent ($M_4 P_1$ in the figure) shall invariably meet the axis at the foot of the first ordinate.



SOLUTION, *by Mr.* HERSCHEL, *the Proposer.*

Let $AP_1 = x$, $P_1\,M_1 = y$, $AP_2 = \varphi(x)$, $\varphi(x)$ being a certain unknown function of x. Then because the point P_3 is determined from P_2 by the same law of construction that P_2 is derived from P_1 we must have $AP_3 = \varphi(AP_2) = \varphi(\varphi(x)) = \varphi^2(x)$, denoting by $\varphi^2(x)$ the repetition of the unknown operation represented by φ. In like manner, if there be more tangents,

$$AP_4 = \varphi\varphi^2(x) = \varphi^3 x, \;\ldots\ldots\ldots\; AP_n = \varphi^{n-1}(x).$$

The next value of AP will be $\varphi^n(x)$, but, by the condition of the problem, the last tangent will fall at the foot of the first ordinate, therefore this distance must be equal to AP_1 or x. Consequently we must have

$$\varphi^n(x) = x.$$

Suppose the form of the function φ to be determined from this *functional equation* the method of doing which we shall presently explain. Then we have $AP_2 = \varphi(x)$, or

$$x - y \cdot \frac{dx}{dy} = \varphi(x);$$

whence separating the variables and integrating

$$\log \frac{y}{c} = \int \frac{dx}{x - \varphi(x)}.$$

All that now remains is to resolve the functional equation $\varphi^n x = x$[*]. The methods pointed out by Mr. Babbage in his ingenious papers on the subject, in the transactions of the royal society 1815, 1816, enable us readily to accomplish this point. Assume $\varphi x = \psi^{-1} f \psi x$, where ψ^{-1} denotes the inverse of the operation denoted by ψ, that is, $\psi^{-1} x$ is such a function that

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[*] The parenthesis in $\varphi^n(x)$ and similar expressions being superfluous, we shall disuse them. The reader will understand that $\varphi^2 x$ denotes $\varphi(\varphi(x))$, $\varphi\psi f x$ stands for $\varphi(\psi(f(x)))$ and so on.

$\psi\psi^{-1}x = x$. For instance, if $\psi x = x^2$, $\psi^{-1}x$ will $= \sqrt{x}$ because $\sqrt{(x)}^2 = x$. Then we shall have

$$\phi^2 x = \phi\phi x = \psi^{-1} f\psi\psi^{-1} f\psi x = \psi^{-1} f^2 \psi x.$$

And in like manner $\phi^n x = \psi^{-1} f^n \psi x$.

Suppose now that fx is any particular solution of $\phi^n x = x$, then we have, $f^n x = x$ and of course $\psi^{-1} f^n \psi x = \psi^{-1} \psi x$. But ψ^{-1} denoting the inverse operation of ψ, there must exist such a form of $\psi^{-1}x$, that shall not only give $\psi\psi^{-1}x = x$, but also $\psi^{-1}\psi x = x$. Thus, in the instance before taken, $\psi^{-1}x$ may be either $+\sqrt{x}$ or $-\sqrt{x}$, because both $(+\sqrt{x})^2$ and $(-\sqrt{x})$ are equal to x but $\psi^{-1}\psi x$ has for its values $+\sqrt{x^2}$ and $-\sqrt{x^2}$, that is $+x$ and $-x$, so that although [$\psi\psi^{-1}x$ is necessarily x, $\psi^{-1}\psi x$ may have more than one form, among which however x is necessarily included. Restricting ψ^{-1} then to this form, we have

$$\psi^{-1} f^n \psi x = x.$$

Consequently if fx satisfy the proposed equation, $\psi^{-1} f\psi x$ will do so likewise, whatever be the form of the function ψ, which is therefore absolutely arbitrary. Now particular solutions of $\phi^n x = x$ readily present themselves. $fx = x.\sqrt{1}$ is obviously one; another may be found by assuming $fx = \frac{a + bx}{c + dx}$ and determining the values of a, b, c, d, and others present themselves without much difficulty, all which, however, will, I believe, be found to be comprised in the formula

$$\phi(x) = \psi^{-1}\left\{\sqrt[n]{1}.\ \psi(x)\right\};$$

though to demonstrate this in any case but when $n = 2$ seems a matter of some difficulty.

If $n = 2$, I have not found any curves in which the geometrical construction of the problem is possible. It is true, by assuming $\varphi x = -x$, $\frac{a^2}{x}$, $\sqrt{a^2 - x^2}$, all which satisfy the equation $\varphi^2 x = x$, we get values of y, which have nothing imaginary in them; thus, if $\varphi x = -x$, we have

$$\log y = \int \frac{dx}{2x} = \frac{1}{2} \log cx, \text{ or } y^2 = cx,$$

the equation of a parabola, but though this equation satisfies the analytical condition the construction is impossible. In fact the subtangent being $2x$, we have $\text{AP}_2 = x - 2x = -x$, to which value of the abscissa these is no corresponding real ordinate. Still however the next value of AP being $-\text{AP}_2$ is equal to x as the problem requires. As this is the first instance I have yet met with where the analytical conditions of a problem are completely satisfied by a real curve, at the same time that the geometrical relations to be fulfilled are eluded by a peculiarity in its form and become imaginary I have thought it worth while to notice it.

If $n = 3$, we shall have better success. One value of φx is $\frac{1}{1-x}$, which gives

$$\log \frac{y}{c} = \int \frac{dx(x-1)}{x^2 - x + 1} = \log \sqrt{x^2 - x + 1} - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right),$$

$\tan^{-1}$ denoting what is usually written thus, arc (tan $=$) This gives

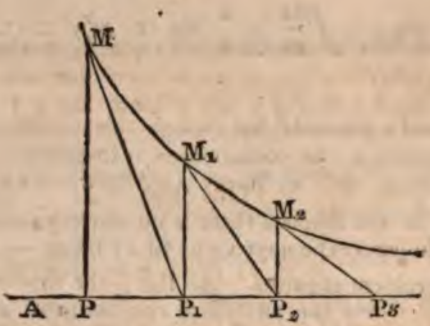
$$y = c \cdot \sqrt{x^2 - x + 1} \cdot e^{-\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}$$

which being always real, whatever real value we assign to x

remains so when for x we substitute $\varphi x = \frac{1}{1-x}$, or $\varphi^2 x = \frac{x-1}{x}$, and therefore affords a satisfactory solution of the problem, though of a somewhat complicated form.

XIX. QUESTION 409, *by Mr.* HERSCHEL.

To determine the nature of a curve such, that any series of tangents being drawn, as in the preceding question, the subtangents



PP_1, P_1P_2, P_2P_3, &c. in the same series shall be all equal, however they may differ in different series originating from a different position of PM.

SOLUTION, *by Mr.* HERSCHEL, *the Proposer.*

Let $AP = x_0$, $AP_1 = x_1$, $AP_2 = x_2$, ... $AP_n = x_n$ also let $PM = y_0$, $P_1M_1 = y_1$ and so on. Then we have

$$AP_{n+1} = AP_n + \frac{y_n\,dx_n}{dy_n}$$

$$\text{or,}\quad \Delta x_n = \frac{y_n\,dx_n}{dy_n}.$$

Now since the subtangent is the same for the points M_n and M_{n+1} therefore this equation is to be integrated on the supposition that $\frac{y_n dx_n}{dy_n}$ is constant, or at least a function of cos $2\pi n$.

$$\text{Let it} = f(\cos 2\pi n). \quad \text{Then}$$

$$x_n = n.\ f(\cos 2\pi n) + \text{C}:$$

but by the same principle that $\frac{y_n dx_n}{dy_n}$ was taken an arbitrary function of cos $2\pi n$, C may be taken equal to another, and therefore to an arbitrary function of the former. Let then $\text{C} = \text{F}f(\cos 2\pi n)$ and we have (writing x and y for x_n and y_n)

$$\frac{ydx}{dy} = f(\cos 2\pi n)$$

$$x = n \cdot f(\cos 2\pi n) + \text{F}f(\cos 2\pi n)$$

and the elimination of n from these gives the differential equation of the curve. To this end we get by substituting for $f(\cos 2\pi n)$ its value,

$$x = n.\ \frac{ydx}{dy} + \text{F}\left(\frac{ydx}{dy}\right);$$

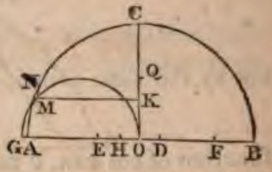
whence,
$$n = \frac{dy}{ydx}\left\{x - \text{F}\left(\frac{ydx}{dy}\right)\right\}$$

and replacing n in the first of the two equations, by this its value

$$\frac{ydx}{dy} = f\left\{\cos 2\pi.\ \frac{dy}{ydx}\left(x - \text{F}\left(\frac{ydy}{pn}\right)\right)\right\}.$$

The general integration of this is of course not to be attempted; even the most particular case except that in which $f(x) = \text{const} = a$, surpasses the powers of analysis; this, however, gives $\frac{dy}{y} = \frac{dx}{a}$, and $y = e^{\frac{x}{a}}$ the equation of a logarithmic curve.

XX. PRIZE QUESTION 410, *by Mr.* LOWRY, *R. M. C.*



Required a geometrical demonstration of the following method of construcing a regular polygon of seventeen sides in a circle: Draw the radius CO at right angles to the diameter AB; on OC and OB, take OQ equal to the half, and OD equal to the eighth part of the radius; make DE and DF each equal to DQ, and EG and FH respectively equal to EQ and FQ; take OK a mean proportional between OH and OQ, and through K, draw KM parallel to AB, meeting the semicircle described on OG in M, draw MN parallel to OC cutting the given circle in N, the arc AN is the seventeenth part of the whole circumference.

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

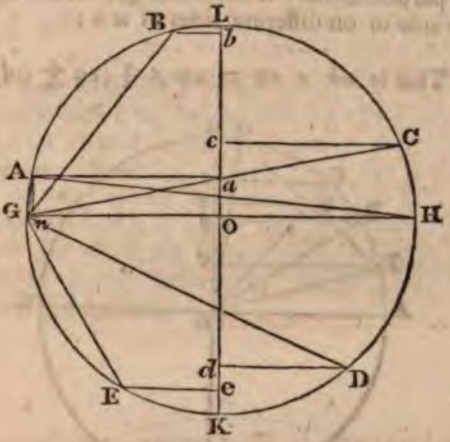
The demonstration of this construction will be shorter and more perspicuous, if the two following *Lemmas* are first demonstrated.

LEMMA I. If the circumference of a circle be divided into any number of equal parts, and perpendiculars be drawn from the points of division to any diameter of the circle; the sum of the perpendiculars drawn on the one side of this diameter is equal to the sum of those on the other side.

Let the circumference of a circle be divided into n equal parts, in the points A, B, C, D, E, &c.; and let Aa, Bb, Cc, &c. be perpendiculars on any diameter KL;

$$\mathrm{A}a + \mathrm{B}b + \mathrm{E}e \text{ \&c.} = \mathrm{C}c + \mathrm{D}d + \text{\&c.}$$

Draw the diameter GH at right angles to KL, and join AG, BG, CG, &c. Also draw An perpendicular to GH and let O be the centre of the circle. Because in the right-angled triangle GAH, $AG^2 = Gn . GH = (GO - Aa) \times 2GO$, therefore



$AG^2 = 2GO^2 - Aa \times 2GO$. In the same way it is shewn that $BG^2 = 2GO^2 - Bb \times 2GO$, $CG^2 = 2GO^2 + Cc \times 2GO$, $DG^2 = 2GO^2 + Dd \times 2GO$, $EG^2 = 2GO^2 - Ee \times 2GO$;

Therefore by addition

$$AG^2 + BG^2 + CG^2 + DG^2 + EG^2 + \&c.)$$

$$= 10GO^2 - 2GO \times (Cc + Dd - Aa - Bb - Ee - \&c.)$$
$$= 2n \times GO^2 + 2GO \times (Cc + Dd + \&c. - Aa - Bb - Ee - \&c.)$$

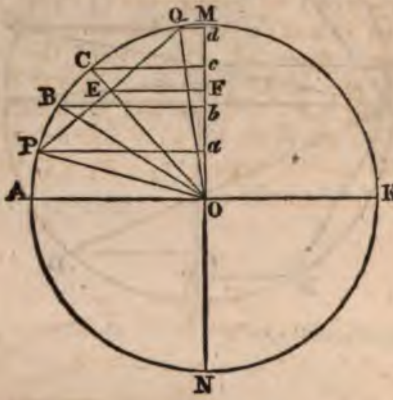
But $AG^2 + BG^2 + CG^2 + DG^2 + EG^2 + \&c. = 2n \times GO^2$, as is well known (see Stewart's General Theorems, Prop. IV.; or Mathematical Repository, first series, vol. 1. p. 19) therefore $2n \times GO^2 = 2n \times GO^2 + 2GO \times (Cc + Dd + \&c. - Aa - Bb - Ee - \&c.)$

and consequently

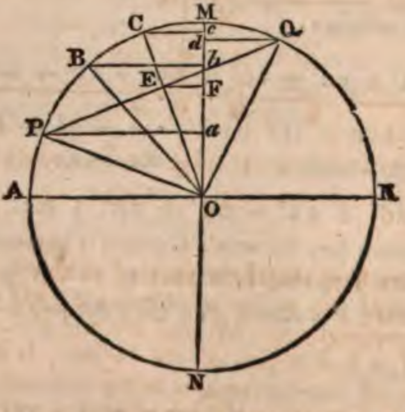
$$2GO \times (Aa + Bb + Ee + \&c.) = 2GO \times (Cc + Dd + \&c.)$$
$$\text{or } Aa + Bb + Ee + \&c. = Cc + Dd + \&c.$$

LEMMA II. Let AK and MN be two diameters of a circle at right angles to one another; AB and AC any two arcs, AQ = their sum and AP = their difference; the rectangle of the perpendiculars from B and C upon MN, is equal to the rectangle contained by the radius of the circle and half the sum or difference of the perpendiculars from P and Q, according as they are on the same side or on different sides of MN:

That is $Bb \times Cc = AO \times \frac{1}{2}(Pa \pm Qd)$.

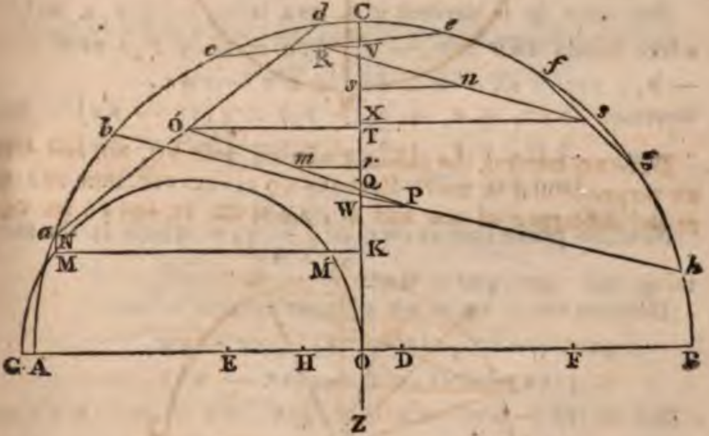


Draw PQ meeting the radius CO in E; join PO, BO, and draw EF perpendicular to MN. Because AQ is = the sum and AP = the difference of AB and AC, AB is = PC and PC = CQ;



therefore CO bisects PQ at right angles in E, and consequently EF is = half the sum or difference of Pa and Qd, namely = to half their sum when P and Q are on the same side of MN, and = to half their difference when they are on different sides of MN. Again, because $AB = PC$, the angle AOB (OBb) = COP; therefore the right-angled triangles BOb, POE are = in all respects; consequently $Bb = OE$, and by similar triangles $OE\,(Bb) : EF\,(\frac{1}{2}(Pa \pm Qd)) :: OC\,(AO) : Cc$; wherefore $Bb \times Cc = AO \times \frac{1}{2}(Pa \pm Qd)$.

To proceed now with the demonstration, let the circumference of the circle ACBZ be divided into 17 equal parts Aa, ab, bc, cd, &c., and let AB and CZ be two diameters drawn at right angles to each other. Bisect ad, which joins the 1st. and 4th. divisions, in O′; bh, which joins the 2nd and 8th, in P; ce, which joins the 3rd. and 5 h. in R; and fg, which joins the 6th and 7th, in S: also bisect the lines O′P and RS in m and n, and from the points of section draw O′T, PW, RV, SX, mr and ns perpendicular to the diameter CZ.



I shall prove first, that the excess of ns above mr is equal to an eighth part of the radius of the circle; and secondly, that each of the rectangles $ns \cdot mr$, $SX \cdot RV$ and $O'T \cdot PW$ is equal to the square of a fourth part of the radius. It is owing to the circumstance of these rectangles being constant quantities that the problem is constructable by plane geometry; and to their equality that the construction is so remarkably simple.

In proving these propositions, it will be necessary to make use of the perpendiculars drawn from the several points of division *a*, *b*, *c*, *d*, &c. upon the diameter CZ; but to prevent confusion in the figure, and to apply the second Lemma with greater facility, it will be better not to draw these perpendiculars, but to represent them by convenient symbols.

Let then P_1, P_2, P_3, P_4, &c. to P_8 represent the perpendiculars drawn from the points *a*, *b*, *c*, &c. upon CO as far as the 8th division; and these taken in the inverse order will also represent the perpendiculars from the 9th, 10th, 11th, &c. divisions as far as the 16th, and the 17th will be the radius AO.

Now first, to prove $ns - mr = \frac{1}{8}\,\text{AO}$, or, which is the same thing, that $8ns - 8mr = \text{AO}$.

It is evident that $2(P_5 + P_6 + P_7 + P_8)$ is the sum of the perpendiculars on the right hand, and $2(P_1 + P_2 + P_3 + P_4) + \text{AO}$ the sum of those on the left of CZ; therefore, by the 1st. Lemma,

$$2(P_5 + P_6 + P_7 + P_8) = 2(P_1 + P_2 + P_3 + P_4) + \text{AO},$$

or $2(P_6 + P_7 + P_5 - P_3) - 2(P_1 + P_4 + P_2 - P_8) = \text{AO}$.

But since *fg* is bisected in S, 2SX is $= P_6 + P_7$; and for a like reason $2\text{RV} = P_3 - P_5$; $2\text{TO}' = P_1 + P_4$; $2\text{PW} = P_8 - P_2$; $2ns = \text{SX} - \text{RV}$, $2mr = \text{O}'\text{T} - \text{PW}$;

therefore $2(P_6 + P_7 + P_5 - P_3) = 4(\text{SX} - \text{RV}) = 8ns$,

$2(P_1 + P_4 + P_2 - P_8) = 4(\text{O}'\text{T} - \text{PW}) = 8mr$,

and $8ns - 8mr = \text{AO}$. .... (*a*)

Next, to prove that $ns\,.\,mr = \left(\frac{\text{AO}}{4}\right)^2$, or which is the same thing, that $4ns\,.\,4mr = \text{AO}^2$.

Because $2ns = \text{SX} - \text{RV}$ and $2mr = \text{O}'\text{T} - \text{PW}$,

$$4ns\,.\,4mr = 4(\text{SX} - \text{RV})\,.\,(\text{O}'\text{T} - \text{PW})$$
$$= 4(\text{SX} - \text{RV})\,.\,\text{O}'\text{T} - 4(\text{SX} - \text{RV})\,.\,\text{PW}.$$

But $4(\text{SX} - \text{RV})\,.\,\text{O}'\text{T}$ is $= 4\text{SX}\,.\,\text{O}'\text{T} - 4\text{RV}\,.\,\text{O}'\text{T}$; and since $2\text{SX} = P_6 + P_7$, $2\text{RV} = P_3 - P_5$ and $2\text{TO}' = P_1 + P_4$,

therefore $4\text{SX}\,.\,\text{O}'\text{T} = (P_6 + P_7)\cdot(P_1 + P_4)$,

and $4\text{RV}\,.\,\text{O}'\text{T} = (P_3 - P_5)\cdot(P_1 + P_4)$.

Now, by the second Lemma,

$$P_6\cdot(P_1 + P_4) = \frac{\text{AO}}{2}\cdot(P_5 + P_7 - P_2 + P_7)$$

$$P_7\cdot(P_1 + P_4) = \frac{\text{AO}}{2}\cdot(P_6 + P_8 - P_3 + P_6);$$

therefore by addition

$$(P_6+P_7)\cdot(P_1+P_4)=\frac{AO}{2}\cdot(2P_6+2P_7+P_8-P_2-P_3+P_4)$$

$$\text{or } 4\mathrm{SX}\cdot\mathrm{O'T}=\mathrm{AO}\cdot(2\mathrm{SX}+\mathrm{PW}-\mathrm{RV}).$$

In the same way it is proved that

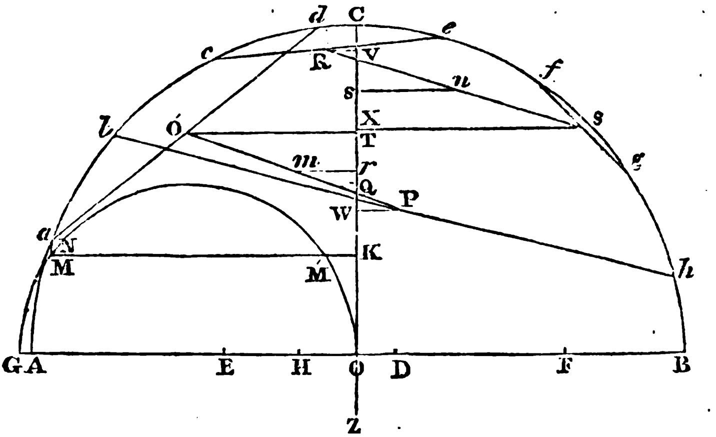
$$4\mathrm{RV}\cdot\mathrm{O'T}=\mathrm{AO}\cdot(2\mathrm{O'T}-\mathrm{SX}-\mathrm{PW}),$$

and taking the difference

$$\begin{aligned}4(\mathrm{SX}-\mathrm{RV})\cdot\mathrm{O'T}&=\mathrm{AO}\cdot(3\mathrm{SX}-2\mathrm{O'T}+2\,\mathrm{PW}-\mathrm{RV})\\&=\mathrm{AO}\cdot(2\mathrm{SX}+2ns-4mr).\end{aligned}$$

Again, because $4(\mathrm{SX}-\mathrm{RV})\cdot\mathrm{PW}=4\mathrm{SX}\cdot\mathrm{PW}-4\mathrm{RV}\cdot\mathrm{PW}$, and $2\mathrm{PW}=P_8-P_2$; therefore, by proceeding as before, $4\mathrm{SX}\cdot\mathrm{PW}=(P_5+P_7)\cdot(P_8-P_2)=\mathrm{AO}\cdot(\mathrm{O'T}-\mathrm{RV}-2\mathrm{PW})$, $4\mathrm{RV}\cdot\mathrm{PW}=(P_3-P_5)\cdot(P_8-P_2)=\mathrm{AO}\cdot(\mathrm{SX}-2\mathrm{RV}-\mathrm{O'T})$, and taking the difference,

$$\begin{aligned}4(\mathrm{SX}-\mathrm{RV})\cdot\mathrm{PW}&=\mathrm{AO}\cdot(2\mathrm{O'T}+3\mathrm{RV}-\mathrm{SX}-2\mathrm{PW})\\&=\mathrm{AO}\cdot(2\mathrm{RV}+4mr-2ns).\end{aligned}$$



Subtracting again, to get the difference of the rectangles, $4(\mathrm{SX}-\mathrm{RV})\cdot\mathrm{O'T}$ and $4(\mathrm{SX}-\mathrm{RV})\cdot\mathrm{PW}$, or $4ns\cdot4mr$, we have

$$4ns\cdot4mr=\mathrm{AO}\cdot(4ns-8mr+2(\mathrm{SX}-\mathrm{RV})).$$

But $2\,(\text{SX} - \text{RV}) = 4ns$; therefore $4ns - 8mr - 2\,(\text{SX} - \text{RV}) = 8ns - 8mr = \text{AO}$, and consequently

$$4ns \cdot 4mr = \text{AO}^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (b)$$

Again, to prove that $\text{SX} \cdot \text{RV} = \left(\frac{\text{AO}}{4}\right)^2$ or that $4\text{SX} \cdot 4\text{RV} = \text{AO}^2$.

$2\text{SX} \cdot 2\text{RV} = (P_6 + P_7) \cdot (P_3 - P_5)$, and, by the second Lemma,

$$P_6 \cdot (P_3 - P_5) = \frac{\text{AO}}{2} \cdot (-P_3 + P_8 - P_1 + P_6)$$

$$P_7 \cdot (P_3 - P_5) = \frac{\text{AO}}{2} \cdot (-P_4 + P_7 - P_2 + P_5)$$

therefore, by addition,

$$2\text{SX} \cdot 2\text{RV} = \frac{\text{AO}}{2} \cdot (P_6 + P_7 + P_8 - P_2 - P_4 - P_1 - P_5 - P_3)$$

$$= \text{AO} \cdot (\text{SX} + \text{PW} - \text{O}'\text{T} - \text{RV}) = \text{AO} \cdot (2ns - 2mr).$$

But $2ns - 2mr = \frac{\text{AO}}{4}$, therefore $2\text{SX} \cdot 2\text{RV} = \text{AO} \times \frac{\text{AO}}{4}$,

and consequently $4\text{SX} \cdot 4\text{RV} = \text{AO}^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (c)$.

Lastly, to prove that $\text{O}'\text{T} \cdot \text{PW} = \left(\frac{\text{AO}}{4}\right)^2$ or that $4\text{O}'\text{T} \cdot 4\text{PW} = \text{AO}^2$.

$2\text{O}'\text{T} \cdot 2\text{PW} = (P_1 + P_4) \cdot (P_8 - P_2)$, and by applying the second Lemma, and proceeding as in the last case, we have

$$2\text{O}'\text{T} \cdot 2\text{PW} = \frac{\text{AO}}{2}\,(P_6 + P_7 + P_8 - P_2 - P_4 - P_1 + P_5 - P_3) = \text{AO} \times \frac{\text{AO}}{4},$$

$$\text{therefore } 4\text{O}'\text{T} \cdot 4\text{PW} = \text{AO}^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (d)$$

These properties being now proved, the demonstration of the construction in the question is readily completed.

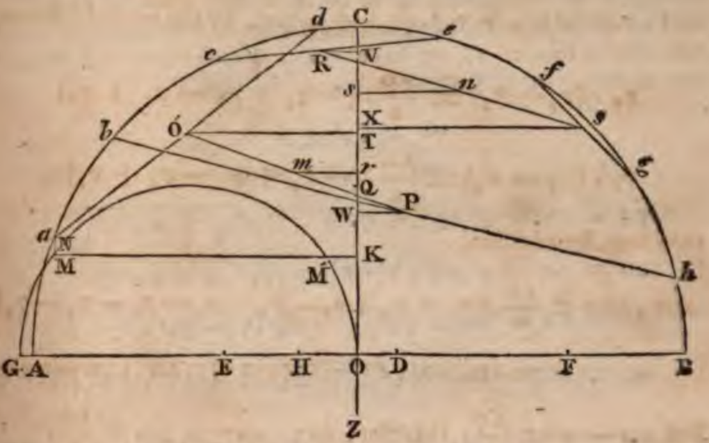
Because $8ns - 8mr = \text{AO}\ (a) = 8\text{OD}$ by construction,

$$ns - mr \text{ is } = \text{OD}; \text{ and } (ns - mr)^2 = \text{OD}^2$$

$$\text{add to these } 4ns \cdot mr = \frac{\text{AO}^2}{4}\ (b)$$

then $(ns + mr)^2 = \mathrm{OD}^2 + \frac{\mathrm{AO}^2}{4} = \mathrm{OD}^2 + \mathrm{OQ}^2 = \mathrm{DQ}^2$ because $\mathrm{OQ} = \frac{\mathrm{OA}}{2}$; therefore $ns + mr = \mathrm{DQ} = \mathrm{DF} = \mathrm{DE}$ by construction, and $ns - mr$ being $= \mathrm{OD}$,

$2ns$ is $= \mathrm{OF}$ and $2mr = \mathrm{OE}$.



But $2ns = \mathrm{SX} - \mathrm{RV}$, therefore $\mathrm{SX} - \mathrm{RV} = \mathrm{OF}$ and $(\mathrm{SX} - \mathrm{RV})^2 = \mathrm{OF}^2$, add $4\mathrm{SX} \cdot \mathrm{RV} = \frac{\mathrm{OA}^2}{4}$ (c);

then $(\mathrm{SX} + \mathrm{RV})^2 = \mathrm{OF}^2 + \frac{\mathrm{OA}^2}{4} = \mathrm{FQ}^2$,

or $\mathrm{SX} + \mathrm{RV} = \mathrm{FQ} = \mathrm{FH}$ by construction;

therefore $2\mathrm{RV} = \mathrm{FH} - \mathrm{OF} = \mathrm{OH}$,

or $\mathrm{P}_3 - \mathrm{P}_5 = \mathrm{OH}$.

Again, because $2mr = \mathrm{O'T} - \mathrm{PW}$, $\mathrm{O'T} - \mathrm{PW} = \mathrm{OE}$ and

$4\mathrm{O'T} \cdot \mathrm{PW} = \frac{\mathrm{OA}^2}{4}$ (d), therefore, as before,

$$(\mathrm{O'T} + \mathrm{PW})^2 = \mathrm{OE}^2 + \frac{\mathrm{AO}^2}{4} = \mathrm{EQ}^2.$$

Consequently $\mathrm{O'T} + \mathrm{PW} = \mathrm{EQ} = \mathrm{EG}$ by construction, and

therefore $2\mathrm{O'T} = \mathrm{OE} + \mathrm{EG} = \mathrm{OG}$,

or $\mathrm{P}_1 + \mathrm{P}_4 = \mathrm{OG}$.

But, by the second Lemma, $\mathrm{P}_1 \cdot \mathrm{P}_4 = \frac{\mathrm{OA}}{2} \cdot (\mathrm{P}_3 - \mathrm{P}_5) = \frac{\mathrm{AO}}{2} \cdot \mathrm{OH} = \mathrm{OQ} \cdot \mathrm{OH} = \mathrm{OK}^2$ by construction; and, by the property of the circle, $\mathrm{KM} + \mathrm{KM'} = \mathrm{OG}$, and $\mathrm{KM} \cdot \mathrm{KM'} = \mathrm{OK}^2$;

therefore KM + KM′ = $P_1 + P_4$ and KM · KM′ = $P_1 \cdot P_4$, and consequently KM = P_1 and KM′ = P_4; that is, KM is = to the perpendicular from *a* upon CO; wherefore the points *a* and N coincide and AN is the seventeenth part of the circumference.

A line drawn through M′ parallel to OC will evidently pass through *d*, the 4th division, and by setting off FQ and EQ on AB both ways, and varying the latter part of the construction a little, any of the other points of division may easily be found.

The calculation from this construction is very easy, for if we take the radius = 1, we have OD = $\frac{1}{8}$ and OQ = $\frac{1}{2}$; whence DE = DF = DQ = $\frac{1}{8}\sqrt{17}$,

$$OF = \tfrac{1}{8}\sqrt{17} + \tfrac{1}{8},$$

$$OE = \tfrac{1}{8}\sqrt{17} - \tfrac{1}{8},$$

$$FH = FQ = \sqrt{(OQ^2 + OF^2)} = \tfrac{1}{8}\sqrt{34 + 2\sqrt{17}},$$

$$EG = EQ = \sqrt{(OQ^2 + OE^2)} = \tfrac{1}{8}\sqrt{34 - 2\sqrt{17}},$$

$$OH = FH - OF = \tfrac{1}{8}\sqrt{34 + 2\sqrt{17}} - \tfrac{1}{8}\sqrt{17} - \tfrac{1}{8},$$

$$OG = EG + OE = \tfrac{1}{8}\sqrt{34 - 2\sqrt{17}} + \tfrac{1}{8}\sqrt{17} + \tfrac{1}{8}.$$

But KM is evidently = $\frac{OG}{2} + \frac{1}{2}\sqrt{(OG^2 - 4OK^2)}$

and $OG^2 = \frac{1}{8^2}(\sqrt{34 - 2\sqrt{17}} + \tfrac{1}{8}\sqrt{17} - \tfrac{1}{8})^2$

$$= \tfrac{1}{64}(52 - 4\sqrt{17} + 2\sqrt{34 - 2\sqrt{17}} \times (\sqrt{17} - 1))$$

$$= \tfrac{1}{64}(52 - 4\sqrt{17} + 8\sqrt{34 + 2\sqrt{17}} - 4\sqrt{34 - 2\sqrt{17}}),$$

by putting $4\sqrt{34 + 2\sqrt{17}}$ for $\sqrt{34 - 2\sqrt{17}} \times (\sqrt{17} - 1)$.

Also $4OK^{\circ} = 4OH \cdot OQ = \frac{1}{4}\sqrt{34 + 2\sqrt{17}} - \frac{1}{4}\sqrt{17} - \frac{1}{4}$;

therefore $OG^2 - 4OK^2 = \frac{1}{16}(17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}})$

and $KM = \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{16}\sqrt{17} - \frac{1}{16} +$

$$\tfrac{1}{8}\sqrt{(17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}})},$$

the same expression that Mr. Leslie has given for the cosine of the 17th part of the circumference, at page 419 of the 2nd edition of his geometry.

End of the first part of the fourth vol.

THE

MATHEMATICAL REPOSITORY,

VOL. IV. PART II.

ORIGINAL ESSAYS.

ARTICLE I.

On Combinations.

By Mr. ROB. J. DISHNEAGH, *Trinity College, Cambridge.*

To the Editor of the Mathematical Repository.

SIR,

I hope the following solution to a problem which occurs in the Novi Commentarii Petropol. Vol. X. will be deemed worthy of insertion in your valuable publication. The proof, if such it can be called, which is given by the Author Josias Weitbrecht is inductive. The advantage of a commodious notation is sufficiently obvious.

I am, &c.

R. J. DISHNEAGH.

June 23, 1814.

The Proposition alluded to, is,

(I). If any limb has n muscles, each having m distinct motions, determine the number of motions when p muscles act at once.

To which he adds another Proposition.

(II). To determine the *total* number of motions which can be exerted by the limb.

For convenience, we shall state the first proposition, thus, Determine the number of combinations of n things $a, b, c,$ &c.

each having m values as a', a'', &c. b', b'' &c.; and supposing p quantities to be taken at once, and no two values of the same quantity occurring in any one combination.

Let, in general, $\varphi^p(n)$ denote the number of combinations of n things each having m different values, the number taken together being represented by the characteristic p, where p and n denote any numbers whatever.

Hence $\varphi^{p-1}(n-1)$ = number of combinations of $n-1$ things each having m values, $p-1$ being taken at once, and no two values of the same quantity occur in any one combination. But each of the m values of a can be prefixed to this, ∴ the number of combinations of n things taken p and p together in which the values of a occur $= m\,\varphi^{p-1}(n-1)$; ∴, setting aside the values of a, we shall have in the same way the number of combinations where b occurs $= m\varphi^{p-1}(n-2)$: and hence evidently, the number of combinations in which c, d, &c. occur will be $= m\,\varphi^{p-1}(n-3)$, $m\,\varphi^{p-1}(n-4)$, &c. respectively. But the whole number of combinations = the sum of all these partial combinations: or

$$\varphi^p(n)=m\varphi^{p-1}(n-1)+m\varphi^{p-1}(n-2)+\ldots\ldots+m\varphi^{p-1}(p-1)$$

$$\therefore \varphi^p(n+1)=m\varphi^{p-1}(n)+m\varphi^{p-1}(n-1)+\ldots\ldots\ldots+m\varphi^{p-1}(p-1)$$

$$\therefore \varphi^p(n+1)-\varphi^p(n)=m\varphi^{p-1}(n) \text{ or } \Delta\varphi^p(n)=m\varphi^{p-1}(n)$$

Hence $\varphi^p(n)=m\Sigma\varphi^{p-1}(n)$ ∴ $\varphi^{p-1}(n)=m\Sigma\varphi^{p-2}(n)$

$\therefore \Sigma\varphi^{p-1}(n)=m\Sigma^2\varphi^{p-2}(n)$ and $\varphi^{p-2}(n)=m\Sigma\varphi^{p-3}(n)$

$\therefore \Sigma^2\varphi^{p-2}(n)=m\Sigma^3\varphi^{p-3}(n)$: &c.

$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

$$\Sigma^{p-2}\varphi^2(n)=m\Sigma^{p-1}\varphi(n).$$

Hence, by multiplying the first two columns *vertically*, we evidently have

$$\varphi^p(n)=m^{p-1}.\Sigma^{p-1}\varphi'(n).$$

Now $\varphi'(n)$=number of combinations of n things each having m distinct values, taken one at once, which is ∴ evidently $= m \times n$

$$\therefore \varphi^p(n)=m^{p-1}.\Sigma^{p-1}mn=m^p\Sigma^{p-1}(n).$$

Now, in integrating successively 1, 2, 3, ...$p-1$ times, we introduce successively the factors $n-1$, $n-2$, ...$n-(p-1)$ and divide successively by 2, 3, 4, ...p

$$\therefore \varphi^p(n)=m^p.\Sigma^{p-1}(n)=m^p.\frac{n(n-1)(n-2)\ldots.(n-(p-1))}{1.2.3\ldots\ldots\ldots p}$$

Corol. $\varphi^p(n) = (p+1)^{th}$ term of the developement of the Binomial $(1+m)^n$.

Prop. 2. Determine the *total* number of combinations of n things each having m values, no two of the same kind occuring in any one combination.

The number evidently $= \varphi'(n) + \varphi^2(n) + \varphi^3(n) + \ldots . \varphi^n(n)$

$$\therefore 1 + \mathrm{N} = 1 + \varphi'(n) + \varphi^2(n) + \varphi^3(n) + \ldots . \varphi^n(n)$$

= sum of all the terms of the developement of $(1+m)^n$ by Corol. to Prop. 1. $= \therefore (1+m)^n$.

$$\therefore \mathrm{N} = (1+m)^n - 1.$$

Corol. If each of the quantities has only one value, or if $m = 1$, then $\mathrm{N} = (1+1)^n - 1 = 2^n - 1$ as is given in Manning's Algebra, and other Books.

ARTICLE II.

On Attractions to Spheroids of small excentricity.

By Mr. Rob. J. Dishneagh, *Trinity College, Cambridge.*

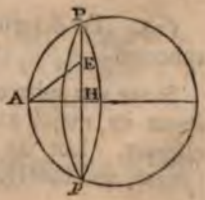
To the Editor of the Mathematical Repository.

Sir,

If the following method of determining the attractions to the poles and equators of spheroids of small excentricity should meet your approbation, I shall be obliged to you for its insertion in the Repository. Bernouilli, in his dissertation on the tides (Prob. VII.) has applied the principle to the solution of the case of the oblate spheroid in which the sections are circles (Prop. 4 infra.): but the conclusions in the other cases he has obtained by a very different process, see Prob. VI. I am, &c.

R. J. Dishneagh.

Lemma. To determine the attraction to any point on the surface of a sphere the attracting force $\propto \frac{1}{\mathrm{D}^2}$.

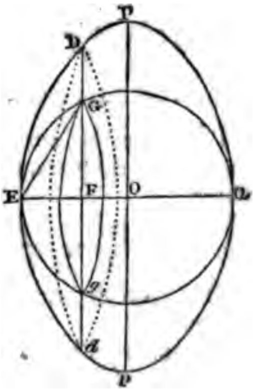


We must first find the attraction of any plane circle Pp on a particle at A. Now let the attraction of any particle E on A be represented by $\frac{1}{\mathrm{AE}^2}$ $\therefore$ its attraction in the direction of the axis = the direct attraction $\times \frac{\mathrm{AH}}{\mathrm{AE}} = \frac{\mathrm{AH}}{\mathrm{AE}^3}$ $\therefore$ the attraction of the circumference of the circle (rad. EH) $= 2\pi\text{EH} \cdot \frac{\text{AH}}{\text{AE}^3}$ $(\pi = 3{\cdot}141592) = 2\pi \cdot \text{AH} \cdot \frac{\text{EH}}{\text{AE}^3}$. Now, calling $\text{AH} = d$, $\text{EH} = v$, ∴ the attraction to the circumference of the circle (rad. EH) $= 2\pi d \cdot \frac{v}{(d^2 + v^2)^{\frac{3}{2}}}$ therefore the fluxion of the attraction to the plane circle equal the attraction to the circumference of the circle $\times \dot{v} = 2\pi d \times \frac{v\dot{v}}{(v^2 + d^2)^{\frac{3}{2}}}$ ∴ the attraction to the plane circle $= 2\pi d \cdot - \frac{1}{\sqrt{(d^2 + v^2)}} + \text{C} = \therefore 2\pi d \left\{\frac{1}{d} - \frac{1}{\sqrt{(d^2 + v^2)}}\right\} = 2\pi \cdot \text{AH} \left\{\frac{1}{\text{AH}} - \frac{1}{\text{AE}}\right\}$ $= 2\pi \left\{1 - \frac{\text{AH}}{\text{AE}}\right\}$ ∴ for the whole circle PQ, the attraction is $= 2\pi \left\{1 - \frac{\text{AH}}{\text{AP}}\right\}$. Now to apply this to determine the attraction to the sphere, let $\text{AH} = x$, radius of the sphere $= r$, ∴ the attraction to the circle $\text{P}p = 2\pi \left\{1 - \frac{x}{\sqrt{(2rx)}}\right\} = 2\pi \left\{1 - \frac{x^{\frac{1}{2}}}{\sqrt{(2r)}}\right\}$ ∴ the fluxion of the attraction to the sphere $= 2\pi \left\{\dot{x} - \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{(2r)}}\right\}$ ∴ the attraction to the sphere $= 2\pi \left\{x - \frac{2}{3} \cdot \frac{x^{\frac{3}{2}}}{\sqrt{(2r)}}\right\}$, and no correction.

Now, for the whole sphere, $x = 2r$, ∴ the attraction to the sphere $= 2\pi \left(2r - \frac{2}{3} \cdot 2r\right) = 2\pi \cdot \frac{2r}{3} = \frac{4\pi r}{3}$.

(1). To determine the attraction to the *equator* of an *oblong* spheroid.

Since the spheroid is generated by the revolution of an ellipse about its major axis $\text{P}p$, the sections $\text{D}d$ parallel to $\text{P}p$ will be ellipses. Let $\text{G}g$ be a corresponding section of the inscribed sphere, and therefore a circle.



Let EO $= r$, PO $= r + \rho$, EF $= x$ and $\pi = 3{\cdot}141592$. Now the area of the circle Gg $= \pi \,.\,$GF$^2 = \pi\,(2rx - x^2)$, and the area of the ellipse D$d =$ the area of the circle $\times \frac{\text{DF}}{\text{GF}} =$ the area of the circle $\times \frac{\text{PO}}{\text{EO}} = \pi\,(2rx - x^2) \times \frac{r+\rho}{r} = \pi\,(2rx - x^2) + \frac{\pi\rho}{r}(2rx - x^2)$.

Now the difference of these areas is $=$ the area of the two meniscuses DG, $gd = \frac{\pi\rho}{r}(2rx - x^2)$........(A)

Now the attraction to these meniscuses $= \frac{\text{number of all the attracting particles}}{(\text{Distance})^2} = \frac{\text{number of attracting particles}}{\text{EG}^2}$, since the surfaces being small each point may be considered equidistant from E, .·. the attraction to the meniscuses $= \frac{\pi\rho}{r} \times (2rx - x^2) \times \frac{1}{\text{EG}^2} = \frac{\pi\rho}{r}(2rx - x^2) \times \frac{1}{2rx} = \frac{\pi\rho}{2r^2} \times (2r - x)$. Now the attraction in the direction of the axis equal the direct attraction $\times \frac{\text{cos.}}{\text{rad.}} =$ the direct attraction $\times \frac{\text{EF}}{\text{EG}} = \frac{\pi\rho}{2r^2} \times (2r - x) \times \frac{x}{\sqrt{(2rx)}} = \frac{\pi\rho}{2r^2\sqrt{(2r)}} \times (2rx^{\frac{1}{2}} - x^{\frac{3}{2}})$.·. the fluxion of the attraction to the solid meniscuses $= \frac{\pi\rho}{2r^2\sqrt{(2r)}} \times (2rx^{\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x})$ theref. the attraction in the direction of the axis $= \frac{\pi\rho}{2r^2\sqrt{(2r)}}\left(\frac{4rx^{\frac{3}{2}}}{3} - \frac{2}{5} \times x^{\frac{5}{2}}\right)$, and no correction necessary.

Hence when $x = 2r$, we have the whole attraction to the difference of the spheroid and the sphere $=$

$$\frac{\pi\rho}{2r^2\sqrt{(2r)}} \times \left\{\frac{4r}{3} \times (2r)^{\frac{3}{2}} - \frac{2}{5} \times (2r)^{\frac{5}{2}}\right\} = \pi\rho\left\{\frac{4}{3} - \frac{4}{5}\right\} = \frac{8\pi\rho}{15} \ldots\ldots\ldots\ldots \text{(B)}.$$

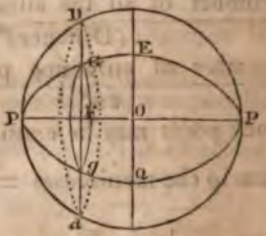
Therefore the attraction to the spheroid = the attraction to the sphere + the attraction to the difference of the spheroid and the sphere

$$= \frac{4\pi r}{3} + \frac{8\pi\rho}{15} = \frac{4\pi r}{3}\left(1 + \frac{2\rho}{5r}\right) \ldots\ldots \ldots\ldots (1).$$

(II). To determine the attraction to the *pole* of an *oblong* spheroid.

The spheroid being generated by the revolution of the ellipse about the major axis Pp, the sections Gg perpendicular to Pp will be circles:

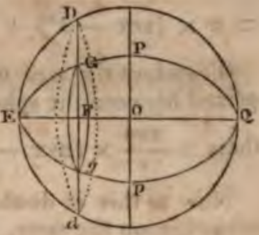
Let Dd be a section of the circumscribed sphere, and ∴ a circle: call PO $= r$, EO $= r - \rho$. Now the area of the circle D$d = \pi \times \text{DF}^2 = \pi \times (2rx - x^2)$ and the area of the circle G$g = \pi \times \text{GF}^2 = \pi \times \text{DF}^2 \times \frac{\text{EO}^2}{\text{PO}^2} = \pi \times (2rx - x^2) \times \left(\frac{r-\rho}{r}\right)^2 = \pi \times (2rx - x^2) \times \left(1 - \frac{2\rho}{r}\right)$ omitting $\frac{\rho^2}{r^2}$. Now the difference of these areas = the area of the sections contained between the sphere and spheroid, ∴ the areas of the two menisci $= \frac{2\pi\rho}{r}(2rx - x^2)$. Now as this is double of (A) and as the remaining part of the investigation is precisely the same, the result will be double of (B). Therefore the attraction to the difference of the sphere and spheroid $= 2 \times \frac{8\pi\rho}{15} = \frac{16\pi\rho}{15}$. Therefore the attraction to the spheroid = the attraction to the sphere − this attraction.



$$= \frac{4\pi r}{3} - \frac{16\pi\rho}{15} = \frac{4\pi r}{3} \times \left(1 - \frac{4\rho}{5r}\right) \ldots\ldots\ldots\ldots \text{(II)}.$$

(III). To determine the attraction to the *equator* of an *oblate* spheroid.

The oblate spheroid being generated by the revolution of the ellipse about the minor axis Pp, all sections Gg parallel to it will be ellipses. Let Dd be a corresponding section of the circumscribed sphere and therefore a circle.



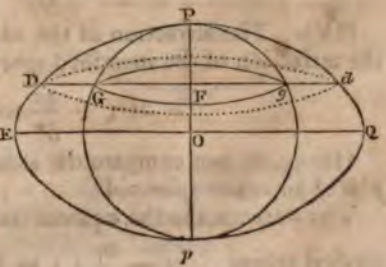
Put $\mathrm{EO} = r$, $\mathrm{PO} = r - \rho$, $\mathrm{EF} = x$. Now the area of the circle Dd $= \pi \times \mathrm{DF}^2 = \pi \times (2rx - x^2)$ and the area of the ellipse GFg = the area of the circle $\times \frac{\mathrm{PO}}{\mathrm{EO}} = \pi \times (2rx - x^2) \times \frac{r-\rho}{r}$ $= \pi \times (2rx - x^2) - \frac{\pi\rho}{r} \times (2rx - x^2)$ ∴ the area of the sections contained between the sphere and spheroid = the difference of these $= \frac{\pi\rho}{r} \times (2rx - x^2)$. Now as this is the same as (A) and as the remaining part of the investigation is the same the conclusion must be the same as (B). Therefore the attraction to the solid menisci comprized between the sphere and spheroid $= \frac{8\pi\rho}{15}$.

Therefore the attraction to the spheroid = the attraction to the sphere — the attraction to the menisci =

$$\frac{4\pi r}{3} - \frac{8r\rho}{15} = \frac{4\pi r}{3} \times \left(1 - \frac{2\rho}{5r}\right) \ldots\ldots\ldots (\mathrm{III}).$$

(IV). To determine the attraction to the *pole* of an *oblate* spheroid.

Since the solid is formed by the revolution of the ellipse about the minor axis Pp, all sections Dd parallel to it will be circles. Let Gg be a corresponding section of the inscribed sphere and ∴ a circle. Let $\mathrm{PO} = r$, $\mathrm{EO} = r + \rho$, $\mathrm{PF} = x$.



Now the area of the circle $\mathrm{G}g = \pi . \mathrm{GF}^2 = \pi . (2rx - x^2)$, and the area of the circle Dd $= \pi \times \mathrm{DF}^2 = \pi \times \mathrm{GF}^2 \times \frac{\mathrm{EO}^2}{\mathrm{PO}^2}$,

$$= \pi \times (2rx - x^2) \times \left(\frac{r + \rho}{r}\right) = \pi \times (2rx - x^2) \times \left(1 + \frac{2\rho}{r}\right)$$

$$= \pi \times (2rx - x^2) + \frac{2\pi\rho}{r} \times (2rx - x^2).$$

Therefore the area of the sections of the solid menisci comprized between the spheroid and the sphere = the difference of these $= \frac{2\pi\rho}{r} \times (2rx - x^2)$.

Now as this is double of (A) and the remaining part of the investigation the same, the result will be double of (B). Therefore the attraction to the solid menisci contained between the spheroid and the sphere will $= \frac{16\pi\rho}{15}$.

Therefore the attraction to the spheroid = the attraction to the sphere + the attraction to the solid menisci

$$= \frac{4\pi r}{3} + \frac{16\pi\rho}{15} = \frac{4\pi r}{3} \times \left(1 + \frac{4\rho}{5r}\right) \ldots\ldots\ldots \text{(IV)}.$$

The four preceding propositions may be thus stated.

(I). The attraction of the *oblong* spheroid on the *equator* is to the attraction of the inscribed sphere

$$\text{as } 1 + \frac{2\rho}{5r} \text{ to } 1.$$

(II). The attraction of the *oblong* spheroid on the *pole* is to the attraction of the circumscribed sphere

$$\text{as } 1 - \frac{4\rho}{5r} \text{ to } 1.$$

(III). The attraction of the *oblate* spheroid on the *equator* is to the attraction of the circumscribed sphere

$$\text{as } 1 - \frac{2\rho}{5r} \text{ to } 1.$$

(IV). The attraction of the *oblate* spheroid on the *pole* is to the attraction of the inscribed sphere

$$\text{as } 1 + \frac{4\rho}{5r} \text{ to } 1.$$

Hence we can compare the attractions to the *equator* and the *pole* of an *oblate* spheroid.

The attraction to the equator is to the attraction to the circumscribed sphere $:: 1 - \frac{2\rho}{5r} : 1$ by (III).

The attraction to the circumscribed sphere is to the attraction to the inscribed sphere $:: r + \rho : r$, by *Lemma.*

The attraction to the inscribed sphere is to the attraction to the pole of the spheroid $:: 1 : 1 + \frac{4\rho}{5r}$.

Therefore, the attraction to the equator of the oblate spheroid is to the attraction to its pole

$$:: \left(1 - \frac{2\rho}{5r}\right)\left(1 + \frac{\rho}{r}\right) : 1 + \frac{4\rho}{5r}.$$

$$:: 1 - \frac{2\rho}{5r} + \frac{\rho}{r} \qquad : 1 + \frac{4\rho}{5r}$$

$$:: 1 + \frac{3\rho}{5r} \qquad : 1 + \frac{4\rho}{5r}.$$

* * *

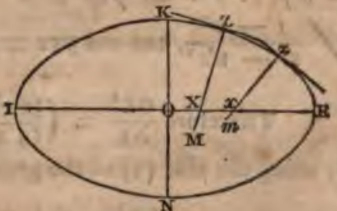
ARTICLE III.

On Finding the Earth's Axes.

By Mr. JAMES ADAMS, *Stone House, Plymouth.*

To find the earth's axes, having given the measure of a degree of a meridian in two given latitudes; the earth being supposed an ellipsoid.

Let IKEN represent a meridian of the earth, KN the less, and IE the greater axis, and Z, *z*, the given latitudes; at right angles to which let ZX, *zx* be drawn, and let ZM, *zm* represent the radii of curvature at Z and *z*.



Put

A = degrees in the greater latitude; S = sine, C = cosine;
B = degrees in the less latitude; *s* = sine, *c* = cosine; rad. 1.
D = length of a degree on the meridian, in latitude A
d = length of a degree on the meridian, in latitude B

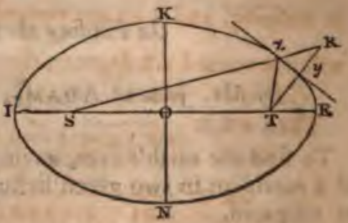
E = an arc, corresponding to the nat. sine $s\sqrt[3]{\frac{d}{\mathrm{D}}} = vs$.

F = an arc, corresponding to the nat. cosine $c\sqrt[3]{\frac{d}{\mathrm{D}}} = vc$.

$57{\cdot}29$, &c. $\times\, \mathrm{D} = \mathrm{R}$; $57{\cdot}29$, &c. $\times\, d = r$; $57{\cdot}29$, &c. being the degrees in a circular arc which is equal to the radius.

Put also $\mathrm{OE}^2 = a^2$, $\mathrm{OK}^2 = b^2$, and L = latus-rectum of the greater axis. Then from the nature of the ellipse $\left(\frac{zx}{\mathrm{ZX}}\right)^2 = \left(\frac{a^2 - (a^2 - b^2)\,\mathrm{S}^2}{a^2 - (a^2 - b^2)\,s^2}\right)^{\frac{3}{2}} = \frac{d}{\mathrm{D}}$, therefore $(1 - v^2)a^2 = (\mathrm{S}^2 - v^2 s^2) \times (a^2 - b^2)$, and thence $b^2 : a^2 :: v^2c^2 - \mathrm{C}^2 : \mathrm{S}^2 - v^2s^2$; or $\frac{\mathrm{OK}^2}{\mathrm{OE}^2} = \frac{(vc + \mathrm{C})\,(vc - \mathrm{C})}{(\mathrm{S} + vs)\,(\mathrm{S} - vs)}$. Hence the ratio of OK to OE is known.

The proportion of the diameters being known, the latus-rectum of the greater axis may be found as follows.



From the foci S, T, draw $\mathrm{S}z$, $\mathrm{T}z$; also draw TR at right angles to the tangent zy, meeting $\mathrm{S}z$ produced in R. Then, by conics $\frac{\mathrm{ST}}{2} = \sqrt{(a^2 - b^2)}$, and $\frac{\mathrm{L}}{2} = \frac{\mathrm{OK}^2}{\mathrm{OE}} = \cos^3 y\mathrm{T}z \times r$. And by Trigonometry, $a : \sqrt{(a^2 - b^2)} :: \sqrt{(\mathrm{S}^2 - v^2s^2)} : \sqrt{(1 - v^2)} :: s : \sin y\mathrm{T}z = s\sqrt{\frac{1 - v^2}{\mathrm{S}^2 - v^2s^2}}$, and $\cos y\mathrm{T}z = \sqrt{\frac{\mathrm{S}^2 - s^2}{\mathrm{S}^2 - v^2s^2}}$;

$$\text{Therefore } \frac{\mathrm{OK}^2}{\mathrm{OE}} = \left(\frac{(\mathrm{S} + s)\,(\mathrm{S} - s)}{(\mathrm{S} + vs)\,(\mathrm{S} - vs)}\right)^{\frac{3}{2}} \times r.$$

By using the sine (S) of the greater latitude we have

$$\frac{\mathrm{OK}^2}{\mathrm{OE}} = \left(\frac{(\mathrm{S} + s)\,(\mathrm{S} - s)}{(\mathrm{S} + vs)\,(\mathrm{S} - vs)}\right)^{\frac{3}{2}} \times v^3 \times \mathrm{R}.$$

But, by Trigonometry,

$$(\mathrm{S} + s) \times (\mathrm{S} - s) = \sin(\mathrm{A} + \mathrm{B}) \times \sin(\mathrm{A} - \mathrm{B}),$$
$$(\mathrm{S} + vs) \times (\mathrm{S} - vs) = \sin(\mathrm{A} + \mathrm{E}) \times \sin(\mathrm{A} - \mathrm{E}),$$
$$(vc + \mathrm{C}) \times (vc - \mathrm{C}) = \sin(\mathrm{A} + \mathrm{F}) \times \sin(\mathrm{A} - \mathrm{F}).$$

Therefore, by substitution,

$$\frac{OK^2}{OE^2}=\frac{(vc+C)(vc-C)}{(S+vs)(S-vs)}=\frac{\sin(A+F)\sin(A-F)}{\sin(A+E)\sin(A-E)},$$

$$\frac{OK^2}{OE}\ \frac{(S+s)(S-s)}{(s+vs)(S-vs)}.r=\left(\frac{s.(A+B)s.(A-B)}{s.(A+E)s.(A-E)}\right)^{\frac{3}{2}}\times 57\text{·}29\&c\times d.$$

And thence,

$$OE=\left(\frac{s.(A+B)s.(A-B)}{s.(A+E)s.(A-E)}\right)^{\frac{1}{2}}\times\frac{s.(A+B)s.(A-B)}{s.(A+F)s.(A-F)}\times 57\text{·}29\&c.\times d,$$

$$OK=\left(\frac{s.(A+B)s.(A-B)}{s.(A+F)s.(A-F)}\right)^{\frac{1}{2}}\times\frac{s.(A+B)s.(A-B)}{s.(A+E)s.(A-E)}\times 57\text{·}29\&c.\times d.$$

$$\frac{OK}{OE}=\left(\frac{\sin(A+F)\sin(A-F)}{\sin(A+E)\sin(A-E)}\right)^{\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

If the earth be a sphere; then $D=d$, and $E=F=B$. Hence $OE=OK=57\text{·}29$, &c. $\times\, d$ ………………….(B).

If one of the places as B, have no latitude, then B and E will be nothing, and F an arc corresponding to the cos $\sqrt[3]{\frac{d}{D}}$; the arc will not exceed six degrees, therefore less than A.

$$\text{Then } OE=\frac{\sin^2 A}{\sin(A+F)\sin(A-F)}\times 57\text{·}29,\ \&c.\times d,$$

$$OK=\frac{\sin A}{\sin(A+F)\sin(A-F)}\times 57\text{·}29,\ \&c.\times d,$$

$$\frac{OK}{OE}=\frac{(\sin(A+F)\sin(A-F))^{\frac{1}{2}}}{\sin A}\ldots\ldots\ldots\ldots(C).$$

If $A=45°$, and $B=0$.

$$\text{Then } OE=\frac{\sin^2 45°}{\sin(45°+F)\cos(45°+F)}\times 57\text{·}29,\ \&c.\times d,$$

$$OK=\frac{\sin 45°}{(\sin(45°+F)\cos(45°+F))^{\frac{1}{2}}}\times 57\text{·}29,\ \&c.\times d.$$

$$\frac{OK}{OE}=\frac{(\sin(45°+F)\cos(45°+F))^{\frac{1}{2}}}{\sin 45°}\ldots\ldots\ldots\ldots(D).$$

If $A=45°$, and B less than A.

hen,

$$\mathrm{OE} = \left(\frac{\sin(45^\circ + \mathrm{B}) \cos(45^\circ + \mathrm{B})}{\sin(45^\circ + \mathrm{E}) \cos(45^\circ + \mathrm{E})}\right)^{\frac{1}{2}} \times \frac{\sin(45^\circ + \mathrm{B}) \cos(45^\circ + \mathrm{B})}{\sin(45^\circ + \mathrm{F}) \cos(45^\circ + \mathrm{F})} \times 57{\cdot}29, \&c \times d,$$

$$\mathrm{OK} = \left(\frac{\sin(45^\circ + \mathrm{B}) \cos(45^\circ + \mathrm{B})}{\sin(45^\circ + \mathrm{F}) \cos(45^\circ + \mathrm{F})}\right)^{\frac{1}{2}} \times \frac{\sin(45^\circ + \mathrm{B}) \cos(45^\circ + \mathrm{B})}{\sin(45^\circ + \mathrm{E}) \cos(45^\circ + \mathrm{E})} \times 57{\cdot}29, \&c. \times d,$$

$$\frac{\mathrm{OK}}{\mathrm{OE}} = \left(\frac{\sin(45^\circ + \mathrm{F}) \cos(45^\circ + \mathrm{F})}{\sin(45^\circ + \mathrm{E}) \cos(45^\circ + \mathrm{E})}\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots(\mathrm{E}).$$

It is plain from the data that E is less than A.

In the Trigonometrical Survey, vol. I. page 300, it is said if
c = cosine and t = tangent of a given latitude,
p = length of the perpendicular degree,
m = length of the meridional degree,
$d = 57{\cdot}29$, &c. the degrees in the circular arc equal to rad.
$r = p - m$, and $\sqrt{\frac{p + rt^2}{m}} = a$. Then,

$$\mathrm{OK} = \frac{dpc}{a^2}\sqrt{(a^2 + t^2)}, \text{ and } \mathrm{EO} = \frac{dpc}{a}\sqrt{(a^2 + t^2)}.\text{[*]}$$

But these expressions may easily be reduced to

$$\mathrm{OK} = \frac{d\sqrt{(pm)} \times c^2}{\left(1 + s\sqrt{\frac{m}{p}}\right)\left(1 - s\sqrt{\frac{m}{p}}\right)},$$

$$\text{and } \mathrm{OE} = \frac{dp \times c}{\left\{\left(1 + s\sqrt{\frac{m}{p}}\right)\left(1 - s\sqrt{\frac{m}{p}}\right)\right\}^{\frac{1}{2}}};$$

s = sine of the given latitude.

Put H = an arc, corresponding to the nat: sin $s\sqrt{\frac{m}{p}}$[†] (rad. unity). Then, per Trigonometry,

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[*] As some of our readers may not have the opportunity of consulting the Trigonometrical Survey, we shall subjoin to this Article, the investigation of the above expressions as given at page 300, vol. I, of that work.

[†] If $s\sqrt{\frac{m}{p}}$, be considered as a *cosine*, then the sine of H must be used.

$$\left(1+s\sqrt{\frac{m}{p}}\right)\left(1-s\sqrt{\frac{m}{p}}\right)=\sin(90^\circ+\mathrm{H})\sin(90^\circ-\mathrm{H})=\cos^2\mathrm{H}.$$

And, by substitution,

$$\mathrm{OK}=d\sqrt{(pm)}\times\frac{\cos^2\mathrm{lat.}}{\cos^2\mathrm{H}};\ \text{and}\ \mathrm{OE}=dp\times\frac{\cos\mathrm{lat}}{\cos\mathrm{H}};$$

Therefore $\frac{\mathrm{OK}}{\mathrm{OE}}=\sqrt{\frac{m}{p}}\times\frac{\cos\mathrm{lat}}{\cos\mathrm{H}}$ ....(F).

If the earth be a sphere; then $p=m$, and $\mathrm{H}=$ lat. Therefore $\mathrm{OK}=\mathrm{OE}=dm$ ....(G).

Example 1. Given $\mathrm{A}=51^\circ\,28'\,40''$; $\mathrm{B}=51^\circ\,5'\,0''$;
$\mathrm{D}=60868$ fath.; $d=60859$ fathoms.

Then (A) ....

$$\log\sqrt[3]{\frac{d}{\mathrm{D}}}=\log\sqrt[3]{\frac{60859}{60868}}=\bar{1}{\cdot}9999786$$

$$\log\sin\text{ of }\mathrm{B}=51^\circ\,5'\,0''\ =\bar{1}{\cdot}8910133$$

$$\log\sin\text{ of }\mathrm{E}=51^\circ\,4'\,47''\,24'''=\bar{1}{\cdot}8909919$$

$$\log\sqrt[3]{\frac{d}{\mathrm{D}}}\ \ldots\ldots\ldots\ldots\ldots\ldots=\bar{1}{\cdot}9999786$$

$$\log\cos 51^\circ\,5'\,0''\ \ldots\ldots\ldots\ldots=\bar{1}{\cdot}7980906$$

$$\log\cos\text{ of }\mathrm{F}=51^\circ\,5'\,8''\,11'''=\bar{1}{\cdot}7980692$$

$\log\sin\mathrm{A}+\mathrm{B}=77^\circ\,26'\,20''\,0'''=9{\cdot}9894786$
$\log\sin\mathrm{A}-\mathrm{B}=0^\circ\,23'\,40''\,0'''=7{\cdot}8378598$
ar; comp. $\mathrm{A}+\mathrm{E}=102^\circ\,33'\,27''\,24'''=0{\cdot}0105155$
ar. comp. $\mathrm{A}-\mathrm{E}=0^\circ\,23'\,52''\,36'''=2{\cdot}1583037$
2) $19{\cdot}9961576$
$9{\cdot}9980788$
$\mathrm{A}+\mathrm{B}$ ....$9{\cdot}9894786$
$\mathrm{A}-\mathrm{B}$ ....$7{\cdot}8378598$
ar. comp. $\mathrm{A}+\mathrm{F}=102^\circ\,33'\,48''\,11'''=0{\cdot}0105253$
ar. comp. $\mathrm{A}-\mathrm{F}=0^\circ\,23'\,31''\,49'''=2{\cdot}1646502$
$57{\cdot}29$, &c. .... $=1{\cdot}7581226$
$d=60859$ ... $=4{\cdot}7843248$
$\mathrm{OE}=3491725$ .... $=6{\cdot}5430401$

Log sin A + B .... = 9·9894786
log sin A — B .... = 7·8378598
ar. comp. A + F .... = 0·0105253
ar. comp. A — F .... = 2·1646502

2) 0·0025139

0·0012569

A + B .... = 9·9894786
A — B .... = 7·8378598
ar. comp. A + E .... = 0·0105155
ar. comp. A — E .... = 2·1583037
57·29, &c. ...... = 1·7581226
d .... = 4·7843248

OK = 3466266 .... = 6·5398619

Log sin A + F .... = 9·9894747
A — F .... = 7·8353498
ar. comp. A + E .... = 0·0105155
ar. comp. A — E .... = 2·1583037

2) 19·9936437

OK ÷ OE = 0·9927086 = 9·9968218

OK : OE :: 0·9927086 : 1 :: 231 : 232·696
:: 1 : 1·0073449 :: 231 : 232·696.

Example 2. Given A = $51^\circ 28' 40''$; B = $50^\circ 41' 0''$,
D = 60868 fath.; d = 60851 fathoms.

Proceeding as before, we find

$$\left.\begin{array}{l} \text{OE} = 3491541 \\ \text{OK} = 3467687 \end{array}\right\} \frac{\text{OK}}{\text{OE}} = \cdot 9931682.$$

Example 3. Given A = $51^\circ 5' 0''$; B = $50^\circ 41' 0''$,
D = 60859, d = 60851.

In this case we find as before,

$$\left.\begin{array}{l} \text{OE} = 3491213 \\ \text{OK} = 3468854 \end{array}\right\} \frac{\text{OK}}{\text{OE}} = \cdot 9935957.$$

Example 4. Given A = $50^\circ 41'$; m = 60851 fathoms,
p = 61182 fathoms; d = 57·29, &c.

Then (F)

$$\begin{array}{lr}
\log \sqrt{\frac{m}{p}} = \sqrt{\frac{60851}{61182}} \ldots\ldots\ldots\ldots = & \bar{1}{\cdot}9988220 \\
\log \sin \text{A} = 50^\circ\ 41'\ 0''\ 0''' \ldots\ldots = & \underline{\bar{1}{\cdot}8885479} \\
\log \sin \text{H} = 50^\circ\ 29'\ 39''\ 11''' \ldots\ldots = & \underline{9{\cdot}8873699} \\
\log \cos \text{A} = 50^\circ\ 41'\ 0''\ 0''' \ldots\ldots = & 9{\cdot}8018192 \\
\log \cos \text{H}\ (\text{a.c.})\ 50^\circ\ 29'\ 39''\ 11''' \ldots\ldots = & \underline{0{\cdot}1964363} \\
 & 9{\cdot}9982555 \\
 & \underline{2} \\
\log \left(\frac{\cos \text{A}}{\cos \text{H}}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots = & 9{\cdot}9965110 \\
\log d = 57{\cdot}29, \&c \ldots\ldots\ldots\ldots\ldots\ldots = & 1{\cdot}7581226 \\
\log \sqrt{(pm)} = \sqrt{(61182 \times 60851)} \ldots = & \underline{4{\cdot}7854457} \\
\text{OK} = 3468002 \ldots\ldots\ldots\ldots\ldots = & \underline{6{\cdot}5400793} \\
\log d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = & 1{\cdot}7581226 \\
\log p = 61182 \ldots\ldots\ldots\ldots\ldots\ldots\ldots = & 4{\cdot}7866237 \\
\log \frac{\cos \text{A}}{\cos \text{H}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots = & \underline{9{\cdot}9982555} \\
\text{OE} = 3491417 \ldots\ldots\ldots\ldots\ldots = & 6{\cdot}5430018 \\
\text{OK} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = & \underline{6{\cdot}5400793} \\
\frac{\text{OE}}{\text{OK}} = 1{\cdot}006752 \ldots\ldots\ldots\ldots\ldots = & \underline{0{\cdot}0029225}
\end{array}$$

Example 5. Given $\text{A} = 50^\circ\,5'$, $\text{P} = 61185$, $m = 60859$, $d = 57{\cdot}29$, &c.

Proceeding as in the last example we get,

$$\left.\begin{array}{l} \text{OK} = 3467940 \\ \text{OE} = 3491400 \end{array}\right\} \frac{\text{OE}}{\text{OK}} = 1{\cdot}006764$$

Example 6. Given $\text{A} = 51^\circ\,28'\,40''$, $\text{P} = 61188$, $m = 60868$, $d = 57{\cdot}29$, &c.

In this case we shall find,

$$\left.\begin{array}{l} \text{OK} = 3468010 \\ \text{OE} = 3491435 \end{array}\right\} \frac{\text{OE}}{\text{OK}} = 1{\cdot}006754.$$

The foregoing six examples with their results, as given in the

Trigonometrical Survey, (vol. 1, page 309) are placed in the following table.

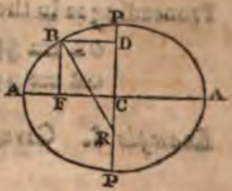
Latitudes.	Degrees on the merid. in fathoms.	Degrees perp. to the meridian in fathoms.	Semi-transverse OE in faths.	Semi-conjugate OK in faths.	Ratio $\frac{OE}{OK}$.
51° 28′ 40″ 51° 5′ 0″	60868 60859	....	3491705	3466266	1·007345
51° 28′ 40″ 50° 41′ 0″	60868 60851	....	3491541	3467687	1·006879
51° 5′ 0″ 50° 41′ 0″	60859 60851	....	3491213	3468854	1·006445
50° 41′ 0″	60851	61182	3491417	3468002	1·006752
51° 5′ 0″	60859	61185	3491400	3467940	1·006764
51° 28′ 40″	60868	61188	3491435	3468010	1·006754
Trigonometrical Survey			3491420	3468007	1·006751

* * *

Extract from vol. 1, page 229, &c. of the Trigonometrical Survey.

Problem. Having the meridional degree, and also the degree perpendicular to the meridian, in a given latitude; to find the earth's axes, supposing it an ellipsoid.

Suppose APAP to be the elliptical meridian passing through the point B whose latitude is given; CA, CP the equatorial and polar semi-axes. Let BF be the ordinate of the point B, and draw BR perpen. to the curve at B, which will be the radius of curvature of the perpendicular degree at that point; also draw BD parallel to AC. Put c and t for the *cosine* and *tangent* of the given latitude; p and m for the lengths of the perpendicular and meridional degrees, respectively; and $d = 57°\cdot29$, &c. the degrees in the circular arc which is equal to the radius.



Then, from the properties of the ellipse, we get FC, or BD $= \frac{CA^2}{\sqrt{(CA^2 + t^2CP^2)}}$; BR $= \frac{CA^2}{c\sqrt{(CA^2 + t^2CP^2)}} = dp$, the radius of curvature of the perpendicular degree; and $\frac{\text{CA}^2 \times \text{CP}^2}{c\sqrt{(\text{CA}^2 + t^2\text{CP}^2)}} = dm$, the radius of curvature of the meridional degree: but the two latter expressions are as 1 to $\frac{\text{CP}^2}{c^2\text{CA}^2 + c^2t^2\text{CP}^2}$; therefore, $1 : \frac{\text{CP}^2}{c^2\text{CA}^2 + c^2t^2\text{CP}^2} :: p : m$; hence, (putting $\frac{1}{t^2+1}$ for c^2, and $r = p - m$), we get $\text{CP}^2 : \text{CA}^2 :: m : p + rt^2$, or $\text{CP} : \text{CA} :: 1 : \sqrt{\frac{p + rt^2}{m}}$, the ratio of the axes.

Let $\sqrt{\frac{p + rt^2}{m}} = a$; then $\text{CA}^2 = a^2\text{CP}^2$ which substituted for CA^2, and we have $\frac{a^2\text{CP}}{c\sqrt{(a^2+t^2)}} = dp$, whence $\text{CP} = \frac{dpc}{a^2} \times \sqrt{(a^2 + t^2)}$; and consequently $\text{CA} = \frac{dpc}{a}\sqrt{(a^2 + t^2)}$.

Corol. 1. If $l =$ the length of a *degree* of *longitude* at the point B, then BD will be its radius of curvature; therefore, *rad.* $: c :: \text{BR} : \text{BD} :: p : l$, hence $p = \frac{l}{c}$, which substituted in the above expressions, we get $\sqrt{\frac{l + crt^2}{cm}} = a$; and $\text{CP} = \frac{dl}{a^2}\sqrt{(a^2 + t^2)}$; and $\text{CA} = \frac{dl}{a}\sqrt{(a^2 + t^2)}$, the semidiameters in this case.

Corol. 2. Because $\frac{\text{CA}^2}{c\sqrt{(\text{CA}^2 + t^2\text{CP}^2)}} = \text{BR} \pm dp$; if b and T represent the *cos.* and *tang.* of some other latitude, and P the *perpendicular degree* in that latitude; then $\frac{\text{CA}^2}{b\sqrt{(\text{CA}^2 + \text{T}^2\text{CP}^2)}} = d\text{P}$ the radius of curvature of P. Hence $\frac{\text{CA}^2}{\text{P}b\sqrt{(\text{CA}^2 + \text{T}^2\text{CP}^2)}} = d$: and the former equation gives $\frac{\text{CA}^2}{pc\sqrt{(\text{CA}^2 + t^2\text{CP}^2)}} = d$; these being equated, we get $\text{P}^2b^2 \times (\text{CA}^2 + \text{T}^2\text{CP}^2) = p^2c^2 \times (\text{CA}^2 + t^2\text{CP}^2)$. Let s, and S, be the *sines* to the *cosines* c, and b; and put $\frac{s^2}{c^2}$ and $\frac{\text{S}^2}{b^2}$ for t^2 and T^2; and we shall have $\text{CA} : \text{CP} :: \sqrt{(p^2s^2}$

$\backsim \text{P}^2s^2) : \sqrt{(\text{P}^2b^2 \backsim p^2c^2)}$, for the ratio of the axes; which being expounded by 1 : n, we have $\text{CA}^2 = n^2\text{CP}^2$, this substituted for CA^2 in the equation $\frac{\text{CA}^2}{pc\sqrt{(\text{CA}^2 + t^2\text{CP}^2)}} = d$, gives $\text{CP} = \frac{dpc}{n^2}\sqrt{(n^2 + t^2)}$; whence $\text{CA} = \frac{dpc}{n}\sqrt{(n^2 + t^2)}$. But the same values for the semi-axes will be obtained by substituting in the other equation.

Hence if l and L be the *degrees* of *longitude* in the given latitudes, we have $p = \frac{l}{c}$, and $\text{P} = \frac{\text{L}}{b}$, which substituted for p, and P, and we shall get the expressions for the semi-axes in that case.

* * *

ARTICLE IV.

Mathematical Scraps,

By Mr. **THOMAS WHITE,** *of the Mathematical School, Dumfries,*

I. *To exhibit the square root of* 2, 3, 5, 7, 11, *&c. in affirmative terms of the powers of* 2.

LEMMA. Whatever a is, we have $\frac{1}{a^n} = 1 + (1 - a) \times \left[\frac{1}{a} + \frac{1}{a^2} + \ldots\ldots + \frac{1}{a^{n-2}} + \frac{1}{a^{n-1}} + \frac{1}{a^n}\right]$. The truth of this will appear by successively transposing the terms of the 2nd side to the first side beginning with the last term. Hence if $a = 2$, $\frac{1}{2^n}$ is $= 1 - \frac{1}{2} - \frac{1}{2^2} - \ldots\ldots - \frac{1}{2^{n-2}} - \frac{1}{2^{n-1}} - \frac{1}{2^n}$; also, $\frac{1}{2^n} - \frac{1}{2^{n+m}} = \frac{1}{2^n}\left(1 - \frac{1}{2^m}\right) = \frac{1}{2^n} \times \left[\frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{m-2}} + \frac{1}{2m-1} + \frac{1}{2^m}\right]$.

Now, $\sqrt{(2)} = \sqrt{(1+1)}$: hence, by the usual process

$1+\frac{1}{2}-\frac{1}{2^2} =$ (by the lem.) $1+\frac{1}{2^2}+\frac{1}{2^3}$; place this below and carry on $1+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\frac{1}{2^{13}}+$ &c. $= \sqrt{(2)}$.

$$\begin{array}{l|l}
 & 1+1 \\
 & 1 \\ \hline
2+\frac{1}{2} & \cdot+1 \\
 & +1+\frac{1}{2^2} \\ \hline
2+1-\frac{1}{2^2} & -\frac{1}{2^2} \\
 & -\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^6} \\ \hline
 & \vdots\ \ \frac{1}{2^3}-\frac{1}{2^6} = \text{(by the lem.)} \\
2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3} & \text{to } \frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6} \\
 & \frac{1}{2^4}+\frac{1}{2^4}+\frac{1}{2^7}+\frac{1}{2^{10}} \\ \hline
 & \cdot\ \frac{1}{2^5}-\frac{1}{2^7}-\frac{1}{2^{10}} = \text{(by the lem.)} \\
2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^7} & \text{to } \frac{1}{2^6}+\frac{1}{2^8}+\frac{1}{2^9}+\frac{1}{2^{10}} \\
 & \frac{1}{2^6}+\frac{1}{2^8}+\frac{1}{2^9}+\frac{1}{2^{11}}+\frac{1}{2^{14}} \\ \hline
 & \frac{1}{2^{10}}-\frac{1}{2^{11}}-\frac{1}{2^{14}} = \text{(by the lem.)} \\
2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\frac{1}{2^{13}} & \text{to } \frac{1}{2^{12}}+\frac{1}{2^{13}}+\frac{1}{2^{14}} \\
 & \frac{1}{2^{12}}+\frac{1}{2^{14}}+\frac{1}{2^{15}}+\frac{1}{2^{17}}+\frac{1}{2^{19}}+\frac{1}{2^{26}} \\ \hline
 & \frac{1}{2^{13}} - \text{\&c. \&c.} \\ \hline
\end{array}$$

In like manner,

$$\sqrt{(3)} = \sqrt{(1+1+1)} = 1+\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^7}+\text{\&c.}$$

$$\sqrt{(5)} = \sqrt{(2^2+1)} = 2 + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^{11}} + \&c.$$

$$\sqrt{(7)} = \sqrt{(2^2+2+1)} = 2 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \&c.$$

$$\sqrt{(11)} = \sqrt{(2^3+2^2+2+1)} = 2 + 1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^8} + \frac{1}{2^{13}} + \&c.$$

$$\sqrt{(13)} = \sqrt{(2^3+2^2+2^0+1)} = 2 + 1 + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^7} + \&c.$$

&c. &c.

Remark. If the digit 3, for example, were to be employed, similar results will be obtained; but, some of the terms will be negative, unless the digit 2 be also admitted. Unity is necessary in all cases, and is equivalent to 2^0, 3^0, &c.

II. *To shew that any equation may be generated, by the addition of its component equations.*

Let the simple equations of a cubic be $x - a = 0$;
$x - b = 0$;
$x - c = 0$.

The sum of the first and second is $2x - (a + b) = 0$; (A)
first and third is $2x - (a + c) = 0$; (B)
second and third is $2x - (b + c) = 0$; (C).

Now, multiply (A) by $\dot{x}$ and take the fluents
$x^2 - (a + b)x +$ correction $= 0$; and
when $x = a$, we have $a^2 - a^2 - ab +$ correction $= 0$;
when $x = b$, $b^2 - ab - b^2 +$ correction $= 0$;
consequently, in both cases the correction is $= + ab$; and,
we have from (A)....$x^2 - (a + b)x + ab = 0$;
in like manner (B)....$x^2 - (a + c)x + ac = 0$;
and from (C)....$x^2 - (b + c)x + bc = 0$.

Again, the sum of these multiplied by $\dot{x}$, gives
$$3x^2\dot{x} - 2(a + b + c)x\dot{x} + (ab + ac + bc)\dot{x} = 0,$$
taking the fluents
$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x + \text{correction} = 0.$$
Now, if $x=a$, $a^3-a^3-ba^2-ca^2+ba^2+ca^2+abc+$correct. $=0$,
if $x=b$, $b^3-ab^2-b^3-cb^2+ab^2+abc+b^2c+$correct. $=0$,
and if $x=c$, $c^3-ac^2-bc^2-c^3+abc+ac^2+bc^2+$correct. $=0$,

therefore in all these cases the correction is $= - abc$; and our derived equation is

$$x^3 - (a + b + c)\,x^2 + (ab + ac + bc)\,x - abc = 0,$$

the same as by Harriott's method: and thus for higher equations.

III. *Concerning equations.*

1°. Suppose a cubic

$$x^3 - (a + b + c)\,x^2 + (ab + ac + bc)\,x - abc = 0,$$

which arises from $(x - a)\,(x - b)\,(x - c) = 0$:

Then, first, $x^3 - abc$ is $= 0$;
and, secondly, $-(a + b + c)\,x^2 + (ab + ac + bc)\,x$ is $= 0$.

Now, nothing can be determined from these latter equations: In their present form, they are paradoxical; and the ambiguity arises from the same symbol x, denoting unequal quantities a, b, and c.

Let then x, x' and x'' denote the roots, to be determined, a, b and c; and we have, by Harriott's method,

$$\left.\begin{array}{l} xx'x'' - ax'x'' + abx'' - abc \\ \qquad\quad - bxx'' + acx' \\ \qquad\quad - cxx' + bcx \end{array}\right\} = 0.$$

Consequently $xx'x'' = abc$, (A)

$$\text{and} \left\{\begin{array}{l} - ax'x'' + abx'' \\ - bxx'' + acx' \\ - cxx' + bcx \end{array}\right\} = 0, \quad (\text{B})$$

In (A) and (B) there is no ambiguity; but, there is a new state of things which seems to indicate that Harriott's theory is not sufficient for simplifying the solution of equations. In (B) there are three unknown quantities x, x' and x''; and apparently one equation only; but, this equation will be fulfilled by splitting it into three equations, viz.

$$- ax'x'' + abx'' = 0;$$
$$- bxx'' + acx' = 0;$$
$$\text{and, } - cxx' + bcx = 0.$$

From these x, x' and x'' are found $= a$, b and c, respectively; and, what is remarkable, equation (A) appears unnecessary: But, the difficulty of applying the considerations suggested by these equations to one of the usual form, $x^3 + px^2 + qx + r = 0$, remains unremoved.

2°. Resuming $(x - a)(x - b)(x - c) = 0$; take the fluxion, we have

$$(x - b)(x - c) + (x - a)(x - c) + (x - b)(x - a) = 0.$$

This will be fulfilled by three quadratics; and, the equation $x^3 - abc = 0$, vanishing, seems to become unnecessary, as before.

Or thus, take the logarithm, we have

$$\log(x - a) + \log(x - b) + \log.(x - c) = 0,$$

the fluxion of which gives $\frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c} = 0$, which is equivalent to three quadratics, and the equation $x^3 - abc = 0$ disappears, as before.

Remark. By taking the fluxion, the roots of the equation continue unchanged, when the resulting equation is resolved into three, notwithstanding the same symbol x remains. Perhaps this observation is new. If we divide the equation which disappears, viz. $xx'x'' - abc = 0$, successively by each of the equations into which (B) is split; and, make each of the remainders $= 0$, relations of the quantities will arise which are consistent with $x = a$, $x' = b$, and $x'' = c$; and, the like may be asserted of the quotients. After all, how are a, b and c to be found from p, q and r? Here is the rub!

IV. *A transformation of equations by which their roots may be approximated, especially when the coefficients and absolute term are considerable.*

Let the cubic $x^3 + px^2 + qx + r = 0$ be supposed $= x + \frac{A}{\alpha + x} + \frac{B}{\beta + x} + C$; then we have

$$x^3 + px^2 + qx + r = \left\{\begin{array}{rcl} x^3 + (\alpha+\beta)\times x^2 & + \alpha\beta x & \\ & + Ax & + \beta A \\ & + Bx & + \alpha B \\ Cx^2 & + (\alpha+\beta)\times Cx & + \alpha\beta C \end{array}\right\} = 0;$$

comparing the homologous terms

$$A = -\frac{r - \alpha q + \alpha^2 p - \alpha^3}{\alpha - \beta},\ B = +\frac{r - \beta q + \beta^2 p - \beta^3}{\alpha - \beta},\ \&\ C = p - \alpha - \beta;$$

and, our equation is now transformed to

$$x + \frac{\alpha^3 - p\alpha^2 + q\alpha - r}{(\alpha + x)(\alpha - \beta)} - \frac{\beta^3 - p\beta^2 + q\beta - r}{(\beta + x)(\alpha - \beta)} + p - \alpha - \beta = 0;$$

in which α and β are assumable and unequal; let $\alpha = 0$, we have

$$x + \frac{r}{\beta x} - \frac{r - \beta q + \beta^2 p - \beta^3}{\beta(\beta + x)} + p - \beta = 0,$$

and β may be any number whatever.

Let then, $\beta = \frac{1}{pqr}$, we have

$$x + \frac{pqr^2}{x} - \frac{r - \frac{1}{pr} + \frac{1}{pq^2r^2} - \frac{1}{p^3q^3r^3}}{\frac{1}{p^2q^2r^2} + \frac{x}{pqr}} + p - \frac{1}{pqr} = 0;$$

and, since p, q and r either are, or may become whole numbers in any equation, the terms $\frac{1}{p^2q^2r^2}$, and $\frac{1}{p^3q^3r^3}$ may be neglected,

which gives $x + \frac{pqr^2}{x} - \frac{pqr^2}{x} - \frac{\frac{1}{qr} - q}{x} + p - \frac{1}{pqr} = 0;$

a quadratic; and, hence,

$$x = \pm \sqrt{\left[\frac{\left(p - \frac{1}{pqr}\right)^2}{4} - \left(q - \frac{1}{qr}\right)\right]} - \frac{\left(p - \frac{1}{pqr}\right)}{2}.$$

Again, let $x^4 + px^3 + qx^2 + rx + s = 0$ be supposed $= x +$

$$\frac{A}{x + \alpha} + \frac{B}{x + \beta} + \frac{C}{x + \nu} + D,$$

the given equation will be transformed to

$$x + \frac{s - \alpha r + \alpha^2 q - \alpha^3 p + \alpha^4}{(\alpha - \beta)(\alpha - \nu)(\alpha + x)} - \frac{s - \beta r + \beta^2 q - \beta^3 p + \beta^4}{(\alpha - \beta)(\beta - \nu)(\beta + x)}$$
$$+ \frac{s - \nu r + \nu^2 q - \nu^3 p + \nu^4}{(\alpha - \nu)(\beta - \nu)(\nu + x)} + p - \alpha - \beta - \nu = 0;$$

where α, β and ν are assumable and unequal: Let $\alpha = 0$, we have $x + \frac{s}{\beta\nu x} + \frac{s - \beta r + \beta^2 q - \beta^3 p + \beta^4}{\beta(\beta - \nu)(\beta - x)}$

$$- \frac{s - \nu r + \nu^2 q - \nu^3 p + \nu^4}{\nu(\beta - \nu)(\nu + x)} + p - \beta - \nu = 0:$$

where β and ν are assumable and unequal. Now, p, q, r, and s, either are, or may become whole numbers. Let $\beta = \frac{1}{pqrs}$; our equation becomes

$$x + \frac{pqrs^2}{\nu x} + \frac{s - \frac{1}{pqs} + \frac{1}{p^2qr^2s^2} - \frac{1}{p^2q^3r^3s^3} + \frac{1}{p^4q^4r^4s^4}}{\frac{1}{p^3q^3r^3s^3} + \frac{x - \nu}{p^2q^2r^2s^2} - \frac{\nu a}{pqrs}}$$

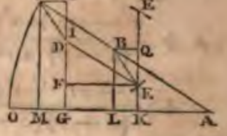
$$- \frac{s - \nu r + \nu^2 q - \nu^3 p + \nu^4}{\left(\frac{\nu}{pqrs} - \nu^2\right)(\nu + x)} + p - \nu - \frac{1}{pqrs} = 0;$$

where ν is assumable: If then ν be found, by trial, nearly, equal to a value of x, in the given equation, we may deem $x - \nu$ as small, and $x + \nu$ as nearly $= 2v$; and since the second and higher powers of $\frac{1}{pqrs}$ may be neglected, our equation becomes

$$x + \frac{r}{x} + \frac{s - \nu r + \nu^2 q - \nu^3 p + \nu^3}{2v^2\left(\nu - \frac{1}{pqrs}\right)} + p - \nu - \frac{1}{pqrs} = 0;$$

a quadratic equation in which ν nearly fulfils the equation given.

V. *About* A, *let a rigid rectilinear rod* ABC *revolve, and let* BCDE *be a jointed parallelogram having given rigid sides; Now supposing the* ∠ D *to move in a vertical right line* GH, *what is the equation of the locus of the* ∠ E ?



On the vertical drop the perpendiculars EF and CH; on the horizontal the perpendicular EK; and let the rod cut the vertical in I. Describe the arc CO. Put AC $= a$; AG $= b$; CD $= c$; DE $= d$; AB $= a - d = e$; GK $= x$; EK $= y$; then DF $= \sqrt{(d^2 - x^2)}$. Since AC is parallel to DE, the triangles DEF, IAG, and CHI are equiangular, $x : d :: b : \frac{bd}{x} =$ AI; therefore $a - \frac{bd}{x} =$ CI; also $x : \sqrt{(d^2 - x^2)} :: b : \frac{b}{x}\sqrt{(d^2 - x^2)} =$ GI; and DG $= y + \sqrt{(d^2 - x^2)}$: Therefore DI $=$ GI $-$ DG $= \frac{b - x}{x}\sqrt{(d^2 - x^2)} - y$; also, $d : \sqrt{(d^2 - x^2)} ::$

$a - \frac{bd}{x} : \frac{ax - bd}{dx}\sqrt{(d^2 - x^2)} =$ HI; and since CD^2 — (DI + IH)^2 = CI^2 — IH^2; we have CD^2 — (DI^2 + 2DI × IH) = CI^2; that is, in symbols,

$$c^2 - \left[\frac{b - x}{x}\sqrt{(d^2 - x^2)} - y\right]^2 - 2\left[\frac{b - x}{x}\sqrt{(d^2 - x^2)} - y\right] \times \frac{ax - bd}{dx}\sqrt{(d^2 - x^2)} = \left(a - \frac{bd}{x}\right)^2,$$

the equation of the curve; whence $y = \frac{e}{d}\sqrt{(d^2 - x^2)} \pm \sqrt{\left[c^2 - \left(\frac{ax}{d} - b\right)^2\right]}$, the required equation. Drop the perpendiculars BL and CM; then GM = CH = $\frac{ax}{d} - b$; DH $= \sqrt{\left[c^2 - \left(\frac{ax}{d} - b\right)^2\right]}$; and BL $= \frac{e}{d}\sqrt{(d^2 - x^2)}$; consequently $y =$ BL ± DH: hence, take QK = BL; and make QE and QE$'$ each = DH; then E and E$'$ are in the required locus.

If x be affirmative, and $\sqrt{\left[c^2 - \left(\frac{ax}{d} - b\right)^2\right]} = 0$, $x = \frac{d}{a}(b + c)$, its greatest value, and $y = e =$ AB.

If x be negative, the above expression becomes $\sqrt{\left[c^2 - \left(\frac{ax}{d} + b\right)^2\right]}$, which, if = 0, gives $-x = \frac{d}{a}(b - c)$, and $y = e$ = AB.

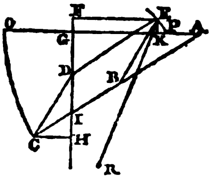
If y be a maximum or minimum, we will find $\frac{e}{a} \cdot \frac{x}{\sqrt{(d^2 - x^2)}}$

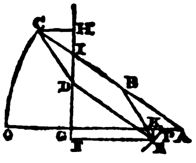
$$= \mp \frac{\frac{ax}{d} - b}{\sqrt{\left[c^2 - \left(\frac{ax}{d} - b\right)^2\right]}};$$

from whence x may be found, and thence the greatest ordinates.

It may be observed that unless b = AG be assumed = AC ∓ CD = $a \mp c$, the branches of the locus cannot be equal on both sides of AG and similarly situated.

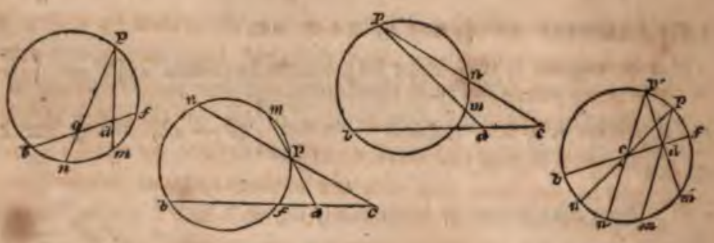
Figures 2nd and 3rd are adapted to AG $= a - c$.





About 14 years ago, a young gentleman connected with Messrs. Bolton and Watt, and once a Pupil of mine, communicated to me the substance of the above problem, and desired to know whether the curve FE (fig. 2 and 3) differed much from a circle: He said that they employed an iron bar ER, which played round a bolt R, to retain the point E in the curve. On receiving the solution which shewed that the curve was not a circle, he observed that the portion of it which was necessary for their purpose was sufficiently small to permit the bar to move without sensible strain on the bolt and gear; and added that the beam AC being 8 feet, the above system permitted a vertical motion of 5 feet, which answered every purpose. Messrs. Bolton and Watt have since improved the contrivance without, however, producing more than a comparatively rectilinear motion vertically.

VI. *Let bf cut a given circle; and c be any point therein; and in it another point d be taken, such, that* $bc \times df = cd^2$ *: Then if any point p be taken in the circumference, and pc and pd be drawn cutting the circle in n and m,* $pd \times dm \times bd$ *is* $= pc \times cn \times cf$.



Demonstration.

By hypoth. $bc : cd :: cd : df$, theref. $bd : cd :: cf : df$; by the circle $pd : df :: bd : dm$, hence $pd : cd :: cf : dm$; by the circle $fc : cn :: pc : cb$,

consequently $pd : dc \times cn :: pc : bc \times dm$:

or, $pd \times dm : nc \times pc :: dc : bc :: df : dc :: cf : bd$:

Hence, we have $pd \times dm \times bd = nc \times cp \times cf$. Q. E. D.

Cor. 1. Hence $pd \times dm \times bc = nc \times cp \times dc$.

2. $pd \times dm \times cd = nc \times cp \times df$.

Several curious properties of the lines concerned may be deduced, and the hope of solving, geometrically, the ancient problem, ‘ the duplication of the cube,’ might seem to revive, by considering the point c in the centre of the circle as in fig. 4 : in this case we have $(bc)^3 = pd \times dm \times bd$; i. e. if $p'dm'$ be perpendicular to bf and $p'cn'$ be drawn, $(bc)^3 = (p'd)^2 \times bd =$ a given parallelopiped ; and the side is to be found of a cube which is = to this solid : But the difficulty lies in determining $p'd$ and bd such, that a circle will pass through b, p' and f so as to make $bc : cd :: cd : df$. This theorem occured to me above 20 years ago : and as it has gotten abroad, I think it proper to publish it myself in your valuable work.

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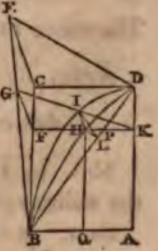
ARTICLE VI.

PROBLEM.

By Mr. PETER NICHOLSON, *Architect, No.* 10, *Oxford Street, London.*

To determine the nature of a surface described by a straight line moving along two other straight lines which are not in the same plane, so that the describing line may be parallel to a plane which is perpendicular to one of the given straight lines, and to find the sections of the surface, as cut by a plane given in position, and also the orthographical projection of the sections.

Let AD and BE be two straight lines not in the same plane. Draw the plane DEC perpendicular to one of the lines AD ; and the plane BEC parallel to AD : draw AB parallel to the plane DEC at such distance from it as to be perpendicular to the plane BEC ; through the straight lines AD and AB draw the plane ABCD which will therefore be a rectangle, perpendicular to both the planes DEC and BEC : let the describing line be KG which will therefore be a line on the surface parallel to the plane DEC, and draw the plane KGF parallel to the plane DEC cutting AD in K, BC in F, and BE in G. In the plane KGF draw HI parallel to FG cutting KG in I and KF at H : then HI will be perpendicular to the plane ABCD.



Let AD or BC $= a$, KF or AB $= b$, CE $= c$, BD $= d$, KH $= x$, BF or QH $= y$, and HI $= z$.

Then, by sim. $\triangle$s, BCE and BFG, BC : CE :: BF : FG;

that is, $$a : c :: y : \frac{cy}{a}.$$

Again, by sim. $\triangle$s, KFG and KHI, KF : FG :: KH : HI;

that is, $$b : \frac{cy}{a} :: x : z.$$

Whence the equation of the surface is $z = \frac{cyx}{ab}$.

Now suppose z a constant quantity, then will $z = \frac{cyx}{ab}$ or $abz = cyx$ be an equation to the hyperbola; therefore the section of the surface cut by a plane parallel to the plane ABCD is an hyperbola.

But to find the projection of the section of the surface cut by any plane upon the plane ABCD, the cutting plane being given in position to the given plane or to the plane ABCD; let BD be the intersection of the cutting plane in the plane ABCD, and let BHD be the projection of the section BID upon the plane ABCD: draw HL perpendicular to BD, cutting BD at L, and join LI, then will the triangles BDA and PHL be similar, for the angles HLP and BAD, are right angles, and because PH is parallel to AB the alternate angles HPL and DBA are equal. Now HL represents the base, HI the perpendicular, and LI the hypothenuse of a right-angled triangle of which LI is an ordinate to the curve in the cutting plane, which will meet the surface at I, and HLI will be the measure of the inclination of the cutting plane to the plane ABCD.

Then, by sim. $\triangle$s, DAB and DKP, DA : AB :: DK : KP,

that is $$a : b :: a - y : \text{KP} = \frac{ba - by}{a}.$$

Therefore $\text{PH} = \text{KH} - \text{KP} = x - \frac{ba - by}{a} = x + \frac{by}{a} - b.$

Again, by sim. $\triangle$s, BDA and PHL, BD : DA :: PH : HL,

that is $$d : a :: x + \frac{by}{a} - b : \text{HL} = \frac{ax + by - ab}{d}.$$

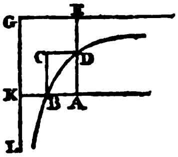
Now let t be the tangent of the angle of inclination HLI to the radius r,

then, $r : t ::$ LH : HI,

that is, $$r : t :: \frac{ax + by - ab}{d} : z = \frac{tax + tby - tab}{rd},$$

which is the equation of the cutting plane; from which take the value of z, and put it equal to the value of z in the equation of the surface gives $\frac{tax + tby - tab}{rd} = \frac{cyx}{ab}$, which is also an equation to the hyperbola, and from which is obtained $y = a^2bt \times \frac{b - x}{ab^2t - cdrx}$.

If x be supposed equal to b, then $y = a^2bt \times \frac{0}{ab^2t - cdrb} = 0$, therefore make AB equal to b, the curve will pass through B.

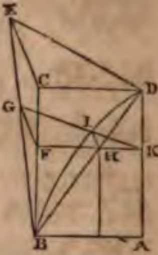


If x be greater then b, the ordinate y will become negative; and will therefore lie on the other side of AB till $cdrx$ become $=$ to ab^2t, then will $y = a^2bt \times \frac{b - x}{0} =$ infinity; therefore make $cdrx = ab^2t$, then $x = \frac{ab^2t}{cdr}$, theref. make AK $= \frac{ab^2t}{cdr}$ and draw GL through K parallel to AD or BC, then GL will be one of the asymptotes of the curve. If $x = 0$, then $y = \frac{a^2b^2t}{ab^2t} = a$; therefore draw AD perpendicular to AB and make AD $= a$ and the curve will pass through D.

Let x be taken negative or on the other side of A, and the signs of the terms in which x are concerned will be changed, then will $y = a^2bt \times \frac{b + x}{ab^2t + cdrx}$.

Lastly, if x be supposed infinite, the terms ab^2t and a^2b^2t will vanish when compared with $cdrx$ and a^2btx, whence $y = \frac{a^2bt}{cdr}$; therefore make AE $= \frac{a^2bt}{cdr}$ and draw EG parallel to AB, GE will be the other asymptote: Therefore in both these positions the curve formed by the surface required and a plane will be an hyperbola, and also its projection upon the plane ABCD.

Let us now suppose the cutting plane to be perpendicular to the plane ABCD.



Then, by sim. $\triangle$s, DAB and DKH,

$$\text{DA} : \text{AB} :: \text{DK} : \text{KH},$$

that is, $a : b :: a-y : x$;

therefore $ax = ba - by$, whence $y = \frac{ab - ax}{b}$.

Therefore, substituting this value of y in the equation of the surface, gives $z = \frac{cyx}{ab} = \frac{cx}{ab}y$ $= \frac{cx}{ab} \times \frac{ab - ax}{b} = \frac{c}{b^2}(bx - x^2)$ which is an equation to the parabola.

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ARTICLE VII.

An expeditious method of ascertaining the Factors of Composite Numbers, and of finding Prime Numbers.

By Mr. P. Barlow, *of the Royal Military Academy.*

It has always been a desideratum with mathematicians to possess a ready method of determining the component factors of composite numbers, and of ascertaining those that are prime. The first attempt at finding prime numbers, that we are acquainted with, was made by Eratosthenes, and gave rise to the invention of what he called his κοσκινον or *sieve*, a particular and interesting account of which is given by Dr. Horsley, in the Philosophical Transactions for 1772. This method however, only applies to the finding of prime numbers in continuation from unity to any required extent, and is of no use in the case of any proposed isolated number. After the work of Diophantus became known in Europe, through the translation of Bachet, and the edition of Fermat, which laid the foundation of our present theory of numbers, a variety of attempts were made for discovering prime numbers by means of certain algebraical formulæ, which should contain those numbers only; and though no such have been found or indeed can be found (as is demonstrated by Le Gendre, "Essai sur la Theorie des Nombres," page 10) yet a few remarkable formulæ were discovered, containing a great many prime numbers in succession, as $x^2 + x + 41$, $2x^2 + 29$, $x^2 + x + 17$, mentioned by Euler, in the Memoirs of Berlin, for 1772.

The first of these formulæ, by making successively $x = 0, 1, 2, 3$, &c. gives a series 41, 43, 47, 53, 61, 71, &c. of which the first forty terms are prime numbers; the second, in the same manner, gives twenty-nine prime numbers, and the third, seventeen. Fermat also asserted that $2^m + 1$ would be always a prime while m was taken any number in the geometrical progression 1, 2, 4, 8, 16, 32, &c. but Euler proved that it fails when $m = 32$. These cases show the danger of drawing conclusions from induction in any mathematical investigation, as there are few cases of this kind, in which we have more reason to infer a general law from particular results than in the first of the formulæ above mentioned.

Waring, in his "Meditationes Algebraicæ," gives a very remarkable theorem relative to prime numbers, of which the discovery he says is due to his friend Sir John Wilson, which is this, "If n be a prime number, then will

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots\cdots (n-1) + 1$$

be divisible by n."

This property belongs exclusively to prime numbers, and therefore offers an infallible method *in abstracto* for ascertaining whether a given number be a prime or composite, but it is unfortunately of no practical utility, in consequence of the enormous magnitude of the product even for a few terms. This elegant theorem was not demonstrated by Waring, but was first proved by La Grange, in the Memoirs of Berlin for 1771.

Such being the difficulty attending the discovery of prime numbers, and of ascertaining the factors of composite numbers, several eminent analysts, particularly Euler, La Grange, and Le Gendre, have endeavoured to abridge the direct operations by investigating the forms of the divisors of numbers; by showing that when a number is of a certain form, it can only have divisors of the same, or certain other forms, whereby the number of tentative divisions are considerably diminished. La Grange, in the Memoirs of Berlin for 1775, has given several very interesting theorems connected with this subject, relative to the divisors of numbers of the form $t^2 + an^2$, t and n being prime to each other; from which it follows, that every number comprised in any one of the forms $t^2 + n^2$, $t^2 + 2n^2$, $t^2 - 2n^2$, $t^2 + 3n^2$, and $t^2 - 5n^2$, can only have for divisors numbers of the same form as themselves, excepting only, in the two latter cases, those divisors that are double an odd number. A great variety of other interesting theorems are deducible from the same principles, and M. Le Gendre has very considerably extended them in his "Essai sur la Theorie des Nombres," and has drawn his results into the form of a table, occupying 44 quarto pages. These formulæ, however, though they offer theoretically a great abridgment of labour, exhibit many beautiful theorems, and display the inexhaustible resourses of the indeterminate analysis, yet it must be acknowledged that they afford little or no advantage in a practical point of view, as it is frequently as difficult to ascertain the form of a number, as it is to discover its factors. M. Gauss also in his "Disquisitiones Arithmeticæ," has given two methods for ascertaining the factors of numbers, but these likewise require prior investigations, and the assistance of extended tables.

Such is the present state of this problem, which the latter author justly calls "one of the most important and most useful in arithtic," and for the solution of which I propose in the following pages, to give a ready practical method, whereby a person, having the slightest knowledge of arithmetic, may discover the factors of composite numbers, or ascertain those that are prime, in about one-tenth of the time that it would employ an expert mathematician to perform the same by the usual operations.

Without referring to the numerical theorems above alluded to, (which as I have observed, would rather tend to lengthen than to abridge the operation, on any number of which the form is not previously known) the only direct method of determining the factors of any proposed number, is to divide it successively by every prime number less than the square root of itself, and if any one of them divide it without a remainder, it is the factor sought; and if, on the contrary, none of them will so divide it, it is a prime. Thus if 9901 were proposed, we must attempt the division of it by every prime number from 1 to 97, the latter being the greatest prime number under $\sqrt{9901}$. The following method is therefore intended to facilitate those divisions, or rather to perform other equivalent operations instead of them, which require only addition, and that, such as may be mentally performed by inspection in the following table, which being intended merely as a specimen, is computed only for numbers under 10000: but a more extended one is given at the conclusion of this article.

The formation of this table depends upon the following obvious property of numbers, viz. if any proposed number be divisible by any other number, the sum of the remainders arising from dividing the several parts of the former by the latter, will be also divisible by that divisor. Thus, if 9911 be divisible by any number *a*, then the sum of the remainders of 9000, of 900, and of 11 divided separately by *a*, will also be divisible by *a*. This property is so obvious as to require no formal demonstration in this place, those who are scrupulous, however, may see the demonstration, at page 12 of my "Theory of Numbers."

Now in the following tablet, the upper line contains all the prime numbers from 7 * to 97, and is therefore marked *primes*; each of the other horizontal lines contains the remainders of the numbers standing on the left hand, when divided by the several prime numbers in the upper line.

Thus the first line after the primes, marked 100, has the numbers 2, 1, 9, 15, 5, &c. shewing that 100 ÷ 7, leaves a remainder 2; by 11, 1; by 13, 9; by 17, 15; &c. The numbers in the second line marked 200, shew the several remainders of 200 when divided by the same prime numbers; and so on, to 9000.

Specimen of a Table for finding the factors of Composite Numbers and for ascertaining Prime Numbers, computed to 10000.

Primes	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
100	2	1	9	15	5	8	13	7	26	18	14	6	47	41	39	33	29	27	21	17	11	3
200	4	2	5	13	10	16	26	14	15	36	28	12	41	23	17	66	58	54	42	34	22	6
300	6	3	1	11	15	1	10	21	4	13	42	18	35	5	56	32	16	8	63	51	33	9
400	1	4	10	9	1	9	23	28	30	31	13	24	29	46	34	65	45	35	5	68	44	12
500	3	5	6	7	6	17	7	4	19	8	27	30	23	28	12	31	3	62	26	2	55	15
600	5	6	2	5	11	2	20	11	8	26	41	36	17	10	51	64	32	16	47	19	66	18
700	7	7	11	3	16	10	4	18	34	3	12	42	11	51	29	30	61	43	68	36	77	21
800	2	8	7	1	2	18	17	25	23	21	26	1	5	33	7	63	19	70	10	53	88	24
900	4	9	3	16	7	3	1	1	12	39	40	7	52	15	46	29	48	24	31	70	10	27
1000	6	10	12	14	12	11	14	8	1	16	11	13	46	56	24	62	6	51	52	4	21	30
2000	5	9	11	11	5	22	28	16	2	32	22	26	39	53	48	57	12	29	25	8	42	60
3000	4	8	10	8	17	10	13	24	3	7	33	39	32	50	11	52	18	7	77	12	63	90
4000	3	7	9	5	10	21	27	1	4	23	1	5	25	47	35	47	24	58	50	16	84	23
5000	2	6	8	2	3	9	12	9	5	39	12	18	18	44	59	42	30	36	23	20	16	53
6000	1	5	7	16	15	20	26	17	6	14	23	31	11	41	22	37	36	14	75	24	37	83
7000	0	4	6	13	8	8	11	25	7	30	34	44	4	38	46	32	42	65	48	28	58	16
8000	6	3	5	10	1	19	25	2	8	5	2	10	50	35	9	27	48	43	21	52	79	46
9000	5	2	4	7	13	7	10	10	9	21	13	23	43	32	33	22	54	21	73	36	11	76

To find the Factors of any proposed number under 10000 *by the above Table.*

It appears from what has been above stated, that if a number, as for example 9911, be divisible by any number a, the sum of the remainders of $9000 \div a$, $900 \div a$, and $11 \div a$, will also be divisible by a. Therefore, to find the factors of any number; add its two last digits to the remainders in the line of hundreds and thousands of that number, and try whether the sum of them (which may be done mentally) be divisible by the prime number immediately above them, if not, try the next two remainders, and so on, till you find one that will divide, or, by going through the whole, show that no one will. In the former case you will have the factor sought, and in the latter, the number is a prime. Thus in the number proposed, viz. 9911, by beginning in the first vertical line, and adding 11 to the remainders in the two lines marked 900, and 9000, we have $4 + 5 + 11 = 20$, which is not divisible by 7; but, in the second column, $9 + 2 + 11 = 22$, is divisible by 11, therefore so also is 9911. If the number proposed had been 9901, it would be found, that 1 added to the sum of each two remainders, is in no case divsible by the corresponding prime number, and therefore 9901 is a prime. Again, let it be proposed to find the factors of 7531, here going over the lines marked 7000, and 500, and adding 31 to the sum of the remainders, we have in the fourth column, $13 + 7 + 31 = 51 = 3 \cdot 17$, which being divisible by 17, this number is also divisible by the same factor.

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* The prime numbers 2, 3, 5, are omitted, because we know immediately whether a given number be divisible by any one of these by mere inspection.

Again, required the factor of 9001; here all that is necessary is to add one successively to each of the remainders in the line marked 9000, and to see whether the sum is in any case equal to the prime number above it; we find upon trial it is not, therefore 9001 is a prime.

There is a trifling inconvenience attending this method, in adding the two remainders together, in consequence of their not being immediately contiguous, the intervening figures in some measure embarrassing the operator; in order to obviate this, I have disposed the above table on five small rods, which may be called *factor rods*, the disposition of the lines being as follows:

Factor Rods. These rods are ivory, five in number, five inches long, and one-fifth of an inch square; on the four sides of the first rod are disposed the four lines of the preceding table marked 100, 200, 300, and 400; on the second, the lines marked 500, 600, 700, and 800; on the third, the lines 900, 1000, 2000, and 3000; on the fourth, the line of *primes*, and the remainders of the lines marked 4000, 5000, and 6000; and on the fifth rod, the line of *primes* again, and the remainders of 7000, 8000; and 9000.

The advantage of this disposition of the numbers is this, that we can bring immediately together the two lines which are to be compared, without any figures intervening, as happens in the table. And as we have two lines of primes, both on the thousands rods, we have always one of them disposable for forming the line of primes. Thus, in the questions already proposed, the rods would be placed as follow, viz. for 9911, we have

Primes	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
9000	5	2	4	7	13	7	10	10	9	21	13	25	43	32	33	22	54	21	73	36	11	76
900	4	9	3	16	7	3	1	1	12	39	40	7	52	15	46	29	48	24	31	70	10	27

The same position of the rods answer for 9901, which is our second example.

For 7531 they are placed thus:

Primes	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	75	79	83	89	97
7000	0	4	6	13	8	8	11	25	7	30	34	44	4	38	46	32	42	65	48	28	58	16
500	3	5	6	7	6	17	7	4	19	8	27	30	23	28	12	31	3	62	26	2	55	15

Here $7 + 13 + 31 = 51 = 17 \cdot 3$, therefore 7531 is divisible by 17.

For 9541 the rods would lay thus:

Primes	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
9000	5	2	4	7	13	7	10	10	9	21	13	23	43	32	33	22	54	21	73	36	11	76
500	3	5	6	7	6	17	7	4	19	8	27	30	23	28	12	31	3	62	26	2	55	15

Here $3 + 5 + 41 = 49 = 7 \cdot 7$, therefore 9541 is divisible by 7.

For 9001, we have

Primes	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
9000	5	2	4	7	13	7	10	10	9	21	13	23	43	32	33	22	54	21	73	36	11	76

It would be useless to multiply examples where the operations are so obvious, but there are one or two instances in which we may abridge still farther the computations, and which it may not be amis to notice.

First. When the proposed number contains in it a 9 in the hundreds place, as 6991. In this case it is better to consider the number under the form 7000—9, and instead of adding 91 to the remainders of 6000 and 900, to subtract 9 from the single remainders of 7000, which is the same thing, or rather look and see whether 9 is one of them; if not, as happens in this case, the number is a prime. In the same manner if 8989 were the proposed number, look for 11 on the remainders of 9000, which being found under 89 shews 89 to be a divisor of 8989.

Secondly. When the two right-hand figures of the proposed number approach towards 100, as in this 9397. Here instead of adding 97 to the lines 9000 and 300, it is easier to subtract 3 from those of 9000 and 400, which is obviously the same thing.

I have at present only alluded to the finding of one factor, but it is evident that the same will apply to finding every factor of the number that is less than 100. Thus in our third example, viz. 9541, besides 3 + 5 + 41 = 49 = 7 × 7, we have also 7 + 10 + 41 = 58 = 2 × 29; and 30 + 23 + 41 = 94 = 2 × 47; therefore 7, 29, 47 are all factors of 9541. If, therefore, in any case, but one factor is found on the rods, the other factor or quotient is a prime number, excepting only when that quotient is again divisible by the same factor.

By this method I have found by repeated trials, that I can find all the factors of any ten numbers under 10000 in about a quarter of an hour, whereas, I have no doubt, that to perform the same in the usual manner, it would require near two hours.

At present I have only spoken of the application of these principles to numbers under 10000, intending this paper merely as a specimen and an illustration of the factor rods, but it is obvious that the same method may be employed to any extent required.

The following table is computed to resolve any number under 100000 into its components factors.

Table for finding of Factors to 100000.

Primes	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	101	103	107	109	113	127	131	137	139
1000	6	10	12	14	12	11	14	8	1	16	11	13	46	56	24	62	6	51	52	4	21	30	91	73	37	19	96	111	83	41	27
2000	5	9	11	11	5	23	28	16	2	32	22	26	39	53	48	57	12	29	25	8	42	60	81	43	74	38	79	95	35	82	54
3000	4	8	10	8	17	10	13	24	3	7	53	39	32	50	11	52	18	7	77	12	63	90	71	13	4	57	62	79	118	123	81
4000	3	7	9	5	10	21	27	1	4	23	1	5	25	47	35	47	24	58	50	16	84	23	61	86	41	76	45	63	70	27	103
5000	2	6	8	2	3	9	12	9	5	39	12	18	18	44	59	42	30	36	23	20	16	53	51	56	78	95	28	47	22	68	135
6000	1	5	7	16	15	20	26	17	6	14	23	31	11	41	22	37	36	14	75	24	37	83	41	26	8	5	11	31	105	109	23
7000	0	4	6	13	8	8	11	25	7	30	34	44	4	38	46	32	42	65	48	28	58	16	31	99	45	24	107	15	57	13	50
8000	6	3	5	10	1	19	25	2	8	5	2	10	50	35	9	27	48	43	21	32	79	46	21	69	82	43	90	126	9	54	77
9000	5	2	4	7	13	7	10	10	9	21	13	23	43	32	33	22	54	21	73	36	11	76	11	39	12	62	78	110	92	95	104
10000	4	1	3	4	6	18	24	18	10	37	24	36	36	29	57	17	60	72	46	40	32	9	1	9	49	81	56	94	44	136	131
20000	1	2	6	8	12	13	19	5	20	33	5	25	19	58	53	34	49	71	13	80	64	18	2	18	98	53	112	61	88	135	123
30000	5	3	9	12	18	8	14	23	30	29	29	14	2	28	49	51	38	70	59	37	7	27	3	27	40	25	55	23	1	134	115
40000	2	4	12	16	5	3	9	10	3	25	10	3	38	57	45	1	27	69	26	77	39	36	4	36	89	106	111	122	45	133	107
50000	6	5	2	3	11	21	4	28	13	21	34	39	21	27	41	18	16	68	72	34	71	45	5	45	31	78	54	89	89	132	99
60000	3	6	5	7	17	16	28	15	23	17	15	28	4	56	37	35	5	67	39	74	14	54	6	54	80	50	110	56	2	131	91
70000	0	7	8	11	4	11	23	2	33	13	39	17	40	26	39	52	65	66	6	31	46	63	7	63	22	22	53	23	46	130	83
80000	4	8	11	15	10	6	18	20	6	9	20	6	23	55	29	2	54	65	52	71	78	72	8	72	71	103	109	117	90	129	75
90000	1	9	1	2	16	1	13	7	16	5	1	42	6	25	25	19	43	64	19	28	21	81	9	81	13	75	52	84	3	128	67

Table for finding of Factors to 100000.

Primes	149	151	157	163	167	173	179	181	191	193	197	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281	283	293	307	311	313
1000	106	94	58	22	165	135	105	95	45	35	15	5	156	108	92	84	68	44	36	247	229	211	193	187	169	157	151	121	79	67	61
2000	63	37	116	44	163	97	31	9	90	70	30	10	101	216	184	168	136	88	72	243	201	159	117	103	61	33	19	242	158	134	122
3000	20	131	107	66	161	59	136	104	135	105	45	15	46	101	49	23	204	132	108	239	173	107	41	19	230	190	170	70	237	201	185
4000	126	74	75	88	159	21	62	18	180	140	60	20	202	209	141	107	39	176	144	235	145	55	234	206	122	66	38	191	9	268	244
5000	83	17	133	110	157	156	167	113	34	175	75	25	147	94	6	191	107	220	180	231	117	3	158	122	14	223	189	19	88	24	305
6000	40	111	34	132	155	118	93	27	79	17	90	30	92	202	98	46	175	25	216	227	89	214	82	38	183	99	57	140	167	91	53
7000	146	54	92	154	153	80	19	122	124	52	105	35	37	87	190	130	10	69	11	223	61	162	6	225	75	256	208	261	246	158	114
8000	103	148	150	13	151	42	124	36	169	87	120	40	193	195	55	214	78	113	47	219	33	110	199	141	244	132	76	89	18	225	175
9000	60	91	51	55	149	4	50	131	23	122	135	45	138	80	157	69	146	157	83	215	5	58	123	57	136	8	227	210	97	292	236
10000	17	34	109	57	147	139	155	45	68	157	150	50	83	188	12	153	214	201	119	211	234	6	47	244	28	165	95	38	176	48	297
20000	34	68	61	114	127	105	131	90	136	121	103	100	166	153	24	77	195	163	238	171	211	12	94	217	56	49	190	76	45	96	281
30000	51	102	13	8	107	7	107	135	13	85	56	150	38	128	36	1	176	125	116	131	188	18	141	190	84	214	2	114	221	144	265
40000	68	136	122	65	87	37	83	180	81	49	9	1	121	83	48	154	157	87	235	91	165	24	188	163	112	98	97	152	90	192	249
50000	85	19	74	22	67	3	59	44	149	13	159	51	104	48	60	78	138	49	113	51	142	30	235	136	140	263	192	190	266	240	233
60000	102	53	26	16	47	142	35	89	23	170	112	101	76	13	72	2	119	11	232	11	119	36	13	109	168	147	4	228	135	288	217
70000	119	87	135	73	27	108	11	134	94	134	65	151	159	201	84	155	100	212	110	222	96	42	60	82	196	31	99	266	4	25	201
80000	136	121	87	130	7	75	166	179	162	98	18	2	31	166	96	79	81	174	229	182	73	48	107	55	224	196	194	11	180	73	185
90000	4	4	39	24	154	40	142	43	39	62	168	52	114	131	108	3	62	136	107	142	50	54	154	28	252	80	6	49	49	121	169

The preceding table is sufficient for ascertaining the factors of any number under 100000, one about three times as large would extend its operation to 1000000. It is proper also to to observe, that the table and principles above explained are equally applicable, whatever method may be first employed for diminishing the number of tentative divisions, whether it be with reference to the forms of divisors according to the principles of La Grange and Le Gendre, or the more extensive exclusions of M. Gauss, for whichever of the above methods we make use of in the first instance, we arrive ultimately at a number of divisional operations, that must be performed; in which cases, it is presumed, the foregoing table, or one carried to the extent indicated above, will become an useful auxilary. When the proposed number is under 10000 or even 100000, the operations are so direct and readily effected by the tables, that it would be useless to attempt any kind of exclusions, but in larger numbers, such a procedure may be had recourse to, and the above principles only employed for essaying the several doubtful divisors.

Hence it appears that these tables may be generally employed, either in conjunction with, or independent of, every other method at present practiced in the solution of this problem, and on that account, may not, perhaps, be thought wholly uninteresting to mathematicians.

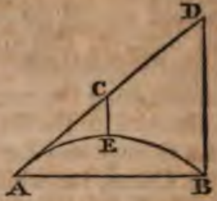
ARTICLE VIII.

ON PROJECTILES.

By Mr. P. BARLOW, *Royal Military Academy.*

1. It has been customary with authors on this subject, to deduce all their results from the properties of the parabola, as the early writers on logarithms drew theirs from the hyperbola and its asymptote; the latter are now, however, wholly excluded from logarithmic investigations, as the former ought doubtless to be from the theory of projectiles, as it serves only to render what is easy in itself, perplexed and intricate: I shall, therefore, in the following article, make no reference to the parabola, except showing that the body in its flight necessarily describes that curve.

2. When a body is projected into space, either, obliquely or parallel to the horizon, it will describe eht curve of a parabola.



Let a body be projected from A in any direction AD; and let AC, AD be the spaces that the body would describe in the times t and T, from the uniform velocity of projection, and CE and DB the spaces through which a heavy body would descend in the same time by gravity; then, by the composition of motions, the body will be found, at the ends of those times, in the points E and B. But, by the laws of uniform motions, we have

$$\text{AC} : \text{AD} :: t : \text{T}, \text{ or } \text{AC}^2 : \text{AD}^2 :: t^2 : \text{T}^2,$$

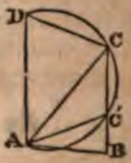
and by those of falling bodies, $\text{CE} : \text{DB} :: t^2 : \text{T}^2$,

whence $\text{CE} : \text{DB} :: \text{AC}^2 : \text{AD}^2$, which is a known property of the parabola; and as the same has place for every point of the projectile's path, that path is a parabola.

PROJECTILES BY GEOMETRY.

3. *To determine the several circumstances of a projectile from geometrical constructions.*

Let AB be the range; AC, the direction of projection, its velocity being v and $16\frac{1}{12} = g$; draw AD perpendicular to the horizon, and equal to $\frac{v^2}{g}$ (or as it is commonly called equal to four times the impetus). Draw BC parallel to AD, meeting AC in C, and join DC.



If now we call t the time of flight, we shall have

$\text{AC} = tv$, $\text{CB} = gt^2$, and $\text{AD} = \frac{v^2}{g}$, therefore,

$\text{AD} : \text{AC} :: \text{AC} : \text{CB}$: consequently, the triangles ADC and ACB are similar, having the angles $\text{DAC} = \text{ACB}$, $\text{ADC} = \text{CAB}$, and $\text{ACD} = \text{ABC}$.

Hence the following constructions.

4. *The velocity and elevation being given, to find the range.*

Take $\text{AD} = \frac{v^2}{g}$, or equal four times the height from which a body must fall to acquire the velocity v; from D, draw DC, making the angle ADC = ABC; from C, draw BC, parallel to AD, meeting the plane in B, and AB will be the range.

5. *The range and elevation being given, to find the velocity.*

Draw BC parallel to DA, or perpendicular to the horizon, meeting AC the direction in C; from C draw CD, making the angle ACD = ABC, so shall AD = $\frac{v^2}{g}$, or, four times the *impetus.*

6. *The range and velocity being given, to find the direction.*

From B draw BC perpendicular to the horizon, and on AD = $\frac{v^2}{g}$, describe a segment capable of containing an angle equal to ABC; which will be cut by BC in the two points C or C′, join AC or AC′, and it will be the direction required.

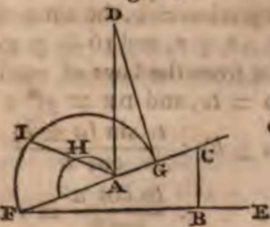
The demonstrations are obvious; for in all these cases, we have AD : AC :: AC : BC.

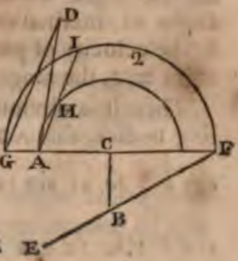
It appears from the last construction, that there are always two different elevations which give the same range, unless in the case where BC becomes a tangent to the segment, in which case there is but one, and that such as to give the maximum range. It is also obvious that in the latter instance, AC bisects the angle DAB, and that in the former, the two lines of direction are equally above and below the line which bisects that angle.

7. In the preceding cases we have supposed the plane to pass through the point of projection, when it does not, the construction is as follows:

Let EF be the plane, AC the direction, and AD = $\frac{v^2}{g}$, as before. Produce AC to meet the plane prolonged in F: Draw DG

Fig. 1.





(fig. 1) making the angle ADG = CFB, on FG describe a semicircle, and concentric with it, another, passing through A, in the latter apply AH = AG, and produce it to meet the other circle in I. Take AC = AI, and draw CB parallel to DA, so shall B be the point on the plane required.

For by similar triangles, AD : AG :: FC : CB,
and by the construction, AG : AI or AC :: IH or GC : FA,
or by composition, AG : AC :: AC : FC,
consequently, AD : AC :: AC : CB,

therefore the projectile passes through B, as shewn in the preceding propositions.

The same construction and demonstration apply to the case in which AC and EF meet each other on the opposite side of DA, as in fig. 2, except that DG is also then drawn on the opposite side of DA, and GC made equal to AI. In this case,

we have AD : AG :: FC : CB,
and by the construction, AG : AC or HI :: GC : FA,
or by division, AG : AC :: AC : FC,

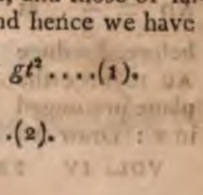
consequently AD : AC :: AC : CB the same as before.

COR. When the plane is parallel to the line of direction, CB is given, and AC is a mean proportional between DA and CB. And when it is perpendicular to it, then AC is given, and CB is a third proportional to DA and AC, as is obvious.

If the point B on the plane be given, all the other circumstances will be determined as in the preceding propositions.

PROJECTILES BY ANALYSIS.

8. Let A be the point, and AD the direction of the projection; AB the plane passing through A, which also represents the range. Let AC be drawn parallel, and BCD perpendicular to the horizon; let the angle CAD of elevation = a, the angle of inclination of the plane CAB = b, the velocity of projection = v, the time of flight = t, the range AB = r, and $16\frac{1}{12} = g$.



Then it is obvious from the laws of motion, and those of falling bodies, that AD = tv, and DB = gt^2; and hence we have

$$\cos b : tv :: \sin(a \pm b) : \frac{tv \sin(a \pm b)}{\cos b} = gt^2 \ldots (1).$$

$$\cos b : tv :: \cos a : \frac{tv \cos a}{\cos b} = r \ldots (2).$$

9. Now to find the greatest height, we must observe that the body will continue to ascend till the velocity of descent from gravity, is equal to the uniform velocity of ascent from projection; that is, calling the time at which the height is the greatest, t', we shall have $2gt' = v \sin a$, or $t' = \frac{v \sin a}{2g}$, but the descent in the time $t' = gt'^2$, in which, substituting the above value of t', it becomes $\frac{v^2 \sin^2 a}{4g}$; and the ascent in the same time from projection is $= t'v \sin a$, or substituting for t' as above, it becomes $\frac{v^2 \sin^2 a}{2g}$; and consequently the difference of these, viz. $\frac{v^2 \sin^2 a}{4g} = h$....(3), the greatest height of the projectile above the point A.

Now, from these three equations (1), (2), and (3), we draw immediately the following, viz.

$$t = \frac{v \sin (a \pm b)}{g \cos b} = \sqrt{\frac{r \sin (a \pm b)}{g \cos a}} = \frac{2 \sin (a \pm b) \sqrt{h}}{g \cos b \,.\, \sin a}$$

$$v = \frac{tg \cos b}{\sin (a \pm b)} = \sqrt{\frac{gr \cos b}{\cos a \,.\, \sin (a \pm b)}} = \frac{2\sqrt{gh}}{\sin a}$$

$$r = \frac{gt^2 \cos a}{\sin (a \pm b)} = \frac{v^2 \cos a \sin (a \pm b)}{g \cos b} = \frac{4h \sin (a \pm b)}{\tan a \,.\, \cos b},$$

$$h = \frac{gt^2 \cos^2 b \,.\, \cos^2 a}{4 \sin^2 (a \pm b)} = \frac{v^2 \sin^2 a}{4g} = \frac{r \tan a \,.\, \cos b}{4 \sin (a \pm b)}.$$

These formulæ involve all the cases of a projectile as far as regard the time, velocity, range, greatest height, angle of elevation, and inclination of the plane, while the latter passes through the point of projection. It is only requisite to observe, that when the plane descends, $(a \pm b)$ becomes $(a + b)$, when it ascends it is $(a - b)$, and when horizontal it is simply a, the angle of the plane being in that case zero. When the angle of the plane is sought, we must introduce the known formulæ,

$$\sin (a \pm b) = \sin a \,.\, \cos b \pm \sin b \,.\, \cos a$$

whence a quadratic will arise, whose two roots will answer to the two different angles of elevation, noticed in the preceding article.

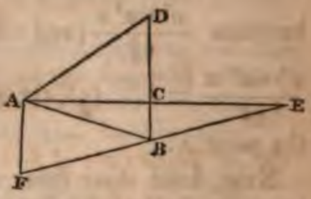
10. If the velocity of the projectile in the curve be required after any time t'', it is obvious that this is compounded of the

constant horizontal velocity $v \cos a$, and the difference of the vertical velocities of ascent from projection $= v \sin a$, and that of descent from gravity $= 2gt''$; that is, calling the required velocity V′, we have

$$V' = \sqrt{\left\{v^2 \cos^2 a + (v \sin a - 2gt')^2\right\}}.$$

11. *When the plane does not pass through the point of projection.*

In this case let the direction of the plane be EF, which being known, the points where it cuts the horizontal and vertical planes EA and AF are also known. Make $AF = m$, $AE = n$, then we have the following proportions, $n : m :: (n - tv \cos a) : \frac{m(n - tv \cos a)}{n} = CB$, to which adding $DE = tv \sin a$, we have



$$DB = gt^2 = tv \sin a + m - \frac{mtv \cos a}{n}, \text{ or}$$

$$t^2 - \left(\frac{v \sin a}{g} - \frac{mv \cos a}{ng}\right) t = \frac{m}{g}.$$

An equation whence the value of t may be found, and consequently also the value of $AB = \sqrt{(AC^2 + BC^2)}$, or

$$AB = \sqrt{\left\{(tv \cos a)^2 + \left(\frac{m(n - tv \cos a)}{n}\right)^2\right\}}.$$

There are six cases to this problem to which the above equation applies.

1 and 2. When the plane rises and meets AC, either produced as at E, or between A and C; in both these m and n are positive.

3. If it descends, and when produced meets the plane AC beyond A, then n is negative and m positive.

4. If it descends, and when produced meets AC between A and C, then m is negative and n positive.

5. When the plane is horizontal, then n is infinite, and the term $\frac{m}{n} tv \cos a$ vanishes, and the equation becomes

$$t^2 - \frac{v \sin a}{g} t = \frac{m}{g}.$$

6. When the plane is vertical, then m is infinite, and all those

terms in the equation in which it does not enter, vanish with respect to those in which it does; and we have, therefore,

$$\frac{mv\cos at}{ng} = \frac{m}{g}, \text{ or } t = \frac{n}{v\cos a}.$$

Whence t in all these cases being determined, the range will be found from the above equation.

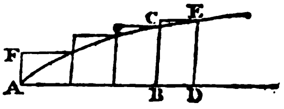
When the point B on the plane is given, then join AB and all the circumstances of velocity, time, elevation, &c. may be computed by the formulæ of art. 9.

ARTICLE IX.

On the Summation of Series which are expressible by a general term.

By Mr. P. BARLOW.

Stirling in his celebrated work "Tractatus de Summatione et Interpolatione Serierum," refers the doctrine of series to that of curves, by conceiving each of the terms to represent the difference of the consecutive ordinates, and consequently the last ordinate as equal to the sum of the terms of the series. The following general method of summation is also best illustrated with reference to a curve, by considering the general term of the series as the ordinate to the curve, and the sum of the series, as consisting of the area of the curve, *plus* the areas of the several exterior parts of the rectangles formed on each ordinate at the common distance of a unit, as in the annexed figure; where the several parallel ordinates are supposed to represent the several terms of the series at the distance of an unit from each other; and consequently the sum of the series, will be denoted by the sum of all those rectangles: which is obviously equal to the area of the curve, *plus* the sum of all the external areas. Let, therefore, $\mathrm{Q}x$ denote the general term of the series, and consequently $f(\mathrm{Q}x)\dot{x} = fx$ the area; then it is obvious, that



$$\mathrm{Q}x - \left\{fx - f(x-1)\right\}$$

will be a general expression for each of the external areas, and therefore calling this last $= (Q'x)$, this will denote the general term of a second series, whose sum is to be obtained in the same manner, and so on.

From these considerations is readily deduced the following general formulæ of summation, viz.

Let the general term of the series be denoted by Qx; and make

$$\int(Qx)\,\dot{x} = fx$$

$$Qx - \left\{fx - f(x-1)\right\} = Q'x,$$

$$\int(Q'x)\,\dot{x} = f'x$$

$$Q'x - \left\{f'x - f'(x-1)\right\} = Q''x,$$

$$\int(Q''x)\,\dot{x} = f''x$$

$$Q''x - \left\{f'x - f''(x-1)\right\} = Q'''x,$$

$$\int(Q'''x)\,\dot{x} = f'''x$$

&c. &c. &c.

So shall $fx + f'x + f''x + f'''x +$ &c. be the sum of the series sought; which will terminate when the series is summable, but continue *ad infinitum* when it is not; its convergency being, however, in this case, much greater than in its original form as will be obvious by refering to the nature of its generation.

I have only time to shew the application of the above principle to one or two simple examples, but which I trust will be sufficient for illustration.

Example 1. Let it be required to sum the series of squares $1^2, 2^2, 3^2, 4^2$ &c.....n^2.

Here the general term $Qx = x^2$,

and $\int(Qx)\,\dot{x} = \int x^2\dot{x} = \frac{1}{3}x^3 = fx$

$$(Qx) - \left\{fx - f(x-1)\right\} = x^2 - \tfrac{1}{3}(x^3 - (x-1)^3) = x - \tfrac{1}{3} = Q'x,$$

$$\int(Q'x)\,\dot{x} = \int(x - \tfrac{1}{3})\,\dot{x} = \tfrac{1}{2}x^2 - \tfrac{1}{3}x = f'x,$$

$$Q'x - \left\{f'x - f'(x-1)\right\} = \tfrac{1}{2} = Q''x$$

$$\int(Q''x)\,\dot{x} = \tfrac{1}{2}x = f''x,$$

Where the operation terminates, and consequently the sum $fx + f'x + f''x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$, which, when x becomes n, is $\dfrac{2n^3 + 3n^2 + n}{6}$.

Example 2. Find the sum of n terms of the series 1^3, 2, 3^3, 4^3, &c.....n^3.

Here the general term $Qx = x^3$,

$$\int(Qx)\dot{x} = \int x^3\dot{x} = \tfrac{1}{4}x^4 = fx,$$

$$Qx - \left\{fx - f(x-1)\right\} = \tfrac{3}{2}x^2 - x + \tfrac{1}{4} = Q'x,$$

$$\int(Q'x)\dot{x} = \tfrac{1}{2}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x = f'x,$$

$$Q'x - \left\{f'x - f'(x-1)\right\} = \tfrac{3}{2}x - 1 = Q''x,$$

$$\int(Q''x)\dot{x} = \tfrac{3}{4}x^2 - x = f''x,$$

$$Q''x - \left\{f''x - f''(x-1)\right\} = \tfrac{3}{4} = Q'''x,$$

$$\int(Q'''x)\dot{x} = \int\tfrac{3}{4}\dot{x} = \tfrac{3}{4}x = f\ x.$$

where the operation again ceases, for if we were now to compute $f^{iv}x$, we should find it equal to zero: therefore by collecting terms, we have

$$fx + f'x + f''x + f'''x = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2$$

$$\text{or (when } x \text{ becomes } n) = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}{}^{2}n.$$

These examples are sufficient for illustrating the principle of summation above given; which, to the best of my knowledge, has not been applied before, to this subject; but, so much has been written on it, that it is difficult to speak positively as to the novelty of any method. I have, however, examined all the most celebrated authors on series, and find nothing that is in any way tantamount to the above principles, yet the simplicity of the conception is such that it is difficult to suppose it has so long escaped the notice of the several ingenious authors, who have treated on the doctrine of series. It must be acknowledged that in its present form, although the principles are simple and obvious, the operations are not so concise as could be wished for, but I am induced to think that the operation exhibited by

$$Qx - \left\{fx - f(x-1)\right\}$$

is reducible to a more ready process than what appears upon the face of the expression, and if this simplification could be effected, I should have no hesitation in stating this method to be by far the most general and simple that has yet been proposed for the summation of series, whose general term is expressed by the same invariable function of x.

* * *

ARTICLE X.

A New Property of the Parabola.

By Lieut. DRUMMOND, *of the Royal Engineers.*

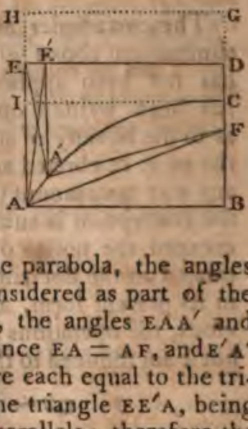
SIR,

The following property of the parabola, which I received from Lieut. Drummond of the Royal Engineers, has not, to the best of my knowledge, been given by any author on the conic sections, and, as it is a very curious one, you will not, perhaps, think it undeserving a place in your valuable Repository.

Your's, &c.

PETER BARLOW.

Let ACB be a parabola, CB the axis, F the focus, ED the directrix; then if the line AF be supposed to revolve about F as a centre, while the line AE moves along the directrix perpendicularly to it, the area generated by the motion of AE, will always be equal to double the area generated by FA; and consequently the whole external area AEGD = double the area ACF.



For draw A′E′ parallel, and indefinitely near, to AE; and draw the diagonals AE′ and A′E; then by the property of the parabola, the angles E′A′A and FA′A are equal, AA′ being considered as part of the tangent at A′; and in the same manner, the angles EAA′ and FAA′ are also equal to each other; and since EA = AF, and E′A′ = A′F; the triangles EAA′ and E′A′A are each equal to the triangle AA′F; but the triangle EAA′ = the triangle EE′A, being on the same base and between the same parallels; therefore the sum of the two triangles EE′A and EA′A, or the quadrilateral space EAA′E′ is double the trilateral space AA′F; and as this is the case in every position of FA′, E′A′, it follows that the whole external area EACD = double the internal area AFC. Q. E. D.

COR. Take DG = FB, and complete the parallelogram DGHE, which is double the triangle ABF; therefore the area ABC = $\frac{1}{2}$ the area HACG, or $\frac{1}{2}$ of the rectangle ABGH, or $\frac{2}{3}$ of the rectangle ABCI, because BC = $\frac{1}{2}$ BG; that is the area of a parabola = $\frac{2}{3}$ of the circumscribing rectangle.

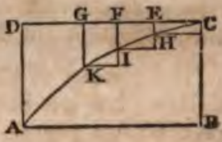
Analytical Demonstration of the same property by Mr. P. BARLOW.

To the above demonstration of Lieut. Drummond's I beg to add the following analytical investigation of the measure of the area of a parabola.

The area of a parabola being exhibited in a manner so extremely simple, most writers on the Conic Sections have wished to demonstrate it in their several works, independently of the doctrine of fluxions, though commonly by an analytical process, but no demonstration I have yet seen is so concise and satisfactory as could be wished for; I beg therefore to propose the following, which appears to me to be some-what preferable in both these respects, to the analytical demonstrations usually given.

To find the area of a Parabola.

Let ABC be a parabola and complete the rectangle ABCD. Make AB $= b$, CB $= a$, and divide DC into any number of equal parts (m) viz. CE, EF, FG, &c. each being therefore $= \frac{b}{m}$. Then by the parabola



$$b^2 : a :: \frac{b^2}{m^2} : \frac{a}{m^2} = \text{EH}$$

$$b^2 : a :: \frac{2^2b^2}{m^2} : \frac{2^2a}{m^2} = \text{FI}.$$

$$b^2 : a :: \frac{3^2b^2}{m^2} : \frac{3^2a}{m^2} = \text{GK},$$

&c. &c.

Multiply each of these by the equal distance $\frac{b}{m}$, and we shall have for the sum of all the rectangles CH, EI, FK, &c.

$$\frac{ab}{m^3} + \frac{2^2ab}{m^3} + \frac{3^2ab}{m^3} + \&c. \frac{m^2ab}{m^3} =$$

$$\frac{ab}{m^3}(1^2 + 2^2 + 3^2 + \&c.\ m^2) =$$

$$\frac{ab}{m^3}\left(\frac{2m^3 + 3m^2 + m}{6}\right) = \frac{2ab}{6} + \frac{3ab}{6m} + \frac{ab}{6m^2}$$

which is true for every possible value of m. But if now we suppose m indefinitely great, so that the breadth of each rectangle, or the distance $\frac{b}{m}$, is indefinitely small; the two last terms of the above expression will vanish; and the first term only $\frac{2ab}{6}$ or $\frac{1}{3}ab$, will be the whole exterior area ADC; and consequently the interior area ABC $= \frac{2}{3}ab$.

ARTICLE XI.

ON CUBIC EQUATIONS.

By Mr. MARK NOBLE, *R. M. College.*

Cubic equations resolved by completing the cube.

"Adjungere jam liceret reductiones æquationum per extractionem surdæ radicis cubicæ, seu et has, ut quæ, perraro utiles sint, brevitatis gratiâ prætereo".

The history of Cardan's rules is familar to learned mathematicians: but no account of the method of the inventors has reached our times. Cardan, who first published the formulæ, contents himself with giving geometrical demonstrations. Of these he was the inventor: but the theorems, he had received on the 25th of March 1539, from Tartaglia, who had discovered rules for the solution of the equations, $x^3 + px^2 = r$, and $x^3 = px^2 + r$, in the year 1530: and on the 12th and 13th of February 1535 he resolved the equations $x^3 + qx = r$, and $x^3 = qx + r$. Tartaglia was not however the first who had found out the general solution of the equation $x^3 + qx = r$: for this, if Cardan may be believed, had been resolved by Scipio Ferreus many years before and the answer communicated to his pupil Antonio Maria Florido, who was contemporary with Tartaglia. Montucla seems to suspect this account, and with justice: Bossut adds that Tartaglia himself, in his violent disputes with Cardan about the invention and publication of these formulæ, denied that Florido was acquainted with the rule. Wallis (Algebra chap. 46, pa. 187, edit. lat.) describes Cardan's demonstration as so perplexed and intricate as to require no small labour to examine it, Cardan going about to elucidate the subject by means of solids expounded on a plane. Vieta, and not Harriot, was the first who thought of substituting $y - \frac{q}{3y}$ for x in the equation $x^3 + qx = r$, which is thus transformed to $y^6 + \frac{q^3}{27} = ry^3$, and may therefore be resolved as a quadratic. (Wallisi, de Algebra Tractatus, cap. 45, Epist. Leibnitii ad Wallisiumi 29, Dec, 1698. Wallisii Opera, Tom. 3, pa. 692. Montucla Hist. Math. Vol. 1, pa. 602.) The same artifice was used by Florimond de Beaune. (Cartesii Geometria, Vol 2, pa. 113, Edit. 1659.) Wallis himself, knowing nothing at all of any of these rules, reading in Oughtred's Clavis, (cap. 18, No. 15, pa. 68, edit. 1648,) that $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$, saw that the solution of the equation $x^3 = qx + r$ would be effected, if he could find u and v such that $3uv = q$, and $u^3 + v^3 = r$, for then would $x = u + v$. This method, which some will conclude could not be unknown to Oughtred, Wallis published in the dedication of his "Adversus Marci Meibomii de proportionibus dialogum, Tractatus, 1656, (Wallisii Opera, Vol. 1, pa. 240, Vol. 2, prop. 135, 185, 186, 187.) From this investigation of the historian of Algebra, that of Hudden, (Cartesii Geometria, vol. 1, page 499, edition 1659,) and that of Newton, (Arithmetica universalis, p. 279, edit. 1722,) do not vary much. This method, the most elegant of any, was extended to biquadratics by Lagrange. Tschirnhausen proposed a general method for the resolution of equations in Act. Erudit. Lipsiæ 1683, this he applied to cubics; and Lagrange to biquadratics. Other substitutions and suppositions may be seen in Waring's Miscellanea Analytica, p. 38, 44, 45, an. 1762. Bezout, Mem. Acad. Sciences, 1762, 1765. Vandremonde, Mem. Acad. Sciences, 1771. Lagrange, Mem. Berol. 1770, 1771. Resolution des équations numeriques, p. 263, ed. 1808. Theorie des Fonctions analytiques, art. 78, pa. 76, edit. 1797. The following method is new to me, and serves to assimilate the solutions of quadratics, cubics, and biquadratics as solved by Lewis Ferrari and Waring, and some particular cases of quadrato-cubics and cubo-cubics (Miscellanea Analytica, p. 35, 36, 37, Meditationes Algebraicæ, pa. 141, 143, edit. 1782. Cartesii Geometria, vol. 1, pa. 490,) to one general principle, viz. completing each side of the equation to a perfect power, such that the exponent of the one power, shall be equal to or a multiple of the exponent of the other.

Let the cubic equation proposed to be resolved be

$$x^3 - qx - r = 0,$$

to each of these equals add $(mx + n)^3$, then,
$(1 + m^3)\,x^3 + 3m^2nx^2 + (3mn^2 - q)x + n^3 - r = (mx + n)^3$, that the side opposite the left hand may be a perfect cube, we must have,

$$mn^2 - \tfrac{1}{3}q = \frac{m^4n^2}{m^3 + 1} \cdots\cdots\cdots\cdots\cdots\cdots (a)$$

and

$$n^3 - r = \frac{m^6n^3}{(m^3 + 1)^2} \cdots\cdots\cdots\cdots\cdots\cdots (b)$$

If these conditions are satisfied, by evolution we obtain,

$$x\sqrt[3]{(m^3 + 1)} + \sqrt[3]{(n^3 - r)} = mx + n,$$

and finally,

$$x = \frac{n - \sqrt[3]{(n^3 - r)}}{\sqrt[3]{(m^3 + 1)} - m}.$$

By (b)

$$n^3 = \frac{r(m^3 + 1)^2}{2m^3 + 1} \cdots\cdots\cdots\cdots\cdots\cdots (c)$$

and

$$n^3 - r = \frac{rm^6}{2m^3 + 1} \cdots\cdots\cdots\cdots\cdots\cdots (d)$$

Again by (a) and (c)

$$\frac{r^2m^3(m^3 + 1)}{(2m^3 + 1)^2} = \frac{q^3}{27} \cdots\cdots\cdots\cdots\cdots\cdots (e)$$

And by (c) (d)

$$x = \sqrt[3]{\frac{r(m^3 + 1)}{2m^3 + 1}} + \sqrt[3]{\frac{rm^3}{2m^3 + 1}}$$

so that we have fallen exactly upon Tartaglia's rule, for the sum of the cubes of the two parts of the value of x is equal to r and by (e) it appears that the product of their cubes is equal to $\frac{1}{27}q^3$: the problem to which it is thus reduced is the 30th. of the 1st. book of Diophantus' Arithmetic. Find therefore the two roots of this quadratic equation $y^2 - ry + \frac{1}{27}q^3 = 0$, let these be y' and y'', then by the theory of equations $y' + y'' = r$ and $y'y'' = \frac{1}{27}q^3$ therefore $x = \sqrt[3]{y'} + \sqrt[3]{y''}$.

The equation $x^3 + px^2 - r = 0$ may be resolved in a manner perfectly similar *without previous reduction:* and indeed the same method may be applied *immediately* to the complete cubic

$$Ax^3 - Bx^2 + Cx - D = 0,$$

ARTICLE XII.

THE BLIND ABBESS.

To the Editor of the Mathematical Repository.

Valeat quanti valet.

SIR,

Every body has heard the story of the good Blind Abbess, and what tricks were put upon her by the knowing sisterhood under her care. But nobody before me, I believe, has been at the pains to discover in how many ways it was possible to vary the cheat. Neither Ozanam nor Montucla seem to have had any idea of the extent to which the joke might be carried on; which convinces me that the story is a dull invention of their own, and never had any foundation in fact. Had such a society of clever ones ever existed, they would have led these frenchmen a fox-chase they little dreamed of.

—'Tis a trifling enquiry.—I know it; and therefore shall not make it appear more trifling by a prosing introduction. Otherwise I might say, that if trifles are not food for the intellect, they are exercise, which is quite as necessary; I might say, that it is no easy matter to settle trifling points in——any doctor in any university can supply the hiatus; I might say that Euler, Waring, Gauss, &c. are never so great as when they trifle; I might say—but bless me, your readers very well know all that might be said about it; so I say no more, except that I am,

Your very obedient servant,

Bath, June, 1816. W. G. HORNER.

The business is to fill the external cells of a square with numbers, as in the margin; in such a manner, that whatever (within certain limits) be the sum of the whole eight numbers, the sum of each three which stand in a row shall constantly be the same. For this I shall give a rule which carries its own demonstration with it, and will conduct us directly to the calculation of the limits and variations.

a	b	c
d		e
f	g	h

RULE. From the whole number s to be disposed of, take double the number n in each row. Separate the remainder into any two parts $b + g$, and again into the same or any other two parts $d + e$. Place b and g, d and e, opposite to each other in the four middle cells. Then, supposing b to be as great or greater than d, e, or g, take any number a, not exceeding $n - b$, and place it in a corner cell adjacent to b. The other corner numbers will follow by subtraction.

Limits. Those of b are obviously 0 and n, inclusively; and the corresponding limits of s are $2n$ and $4n$. And since those of a are 0 and $n - b$, the four corner numbers may undergo $n - b + 1$ changes, while b, d, e, g, remain unaltered.

Variations. 1. When $b = d = e = g$, the variations, as deduced from what was just said, will form a series whose general term is $n - b + 1$; b varying from 0 to n. The sum of this series will be $\frac{1}{2}(n + 1 . n + 2)$.

2. If $s - 2n$ were resolved into two identical pairs of unequal numbers, or if $b = d = 1 \ldots . n$, and $e = g = 0 \ldots . b - 1$; then, since a has $n - b + 1$ values, and $e = g$ has b values, and each assumption for b and e admits 4 varieties of collocation, viz. $\begin{array}{ccc} & b & \\ b & & e \\ & e & \end{array}$, $\begin{array}{ccc} & e & \\ b & & e \\ & b & \end{array}$, $\begin{array}{ccc} & b & \\ e & & b \\ & e & \end{array}$, $\begin{array}{ccc} & e & \\ e & & b \\ & b & \end{array}$; the general term expressing the number of variations is $4b(n - b + 1)$. Consequently the sum is $\frac{2}{3}(n . n + 1 . n + 2)$.

3. Let one pair consist of equal, and the other of unequal numbers; or $d = e = \frac{1}{2}(b + g) = \frac{1}{2}b \ldots . b - 1$, or $\frac{1}{2}(b + 1) \ldots . b - 1$; the former limits obtaining when b and g are even, and the latter when they are odd. Then d will have $\frac{1}{2}b$ or $\frac{1}{2}(b - 1)$ values respectively. And since here also there are four distinct collocations, the general terms will be respectively $2b(n - b + 1)$ and $2(b - 1) . (n - b + 1)$.

If $n = 2m$, while b is alternately 2μ and $2\mu - 1$, μ representing the variable whose limits are 1 and m, the sum of the series whose general terms we have just given, will be $s(4\mu . 2m - 2\mu + 1) + s(2 . 2\mu - 2 . 2m - 2\mu + 2) = -16s\mu^2 + (16m + 20)s\mu - (8m + 8)s\mu^0 = \frac{8}{3}m^3 + 2m^2 - \frac{2}{3}m = \frac{1}{6}(2n^3 + 3n^2 - 2n)$, when n is an even number.

And, if $n = 2m + 1$, while b is alternately 2μ and $2\mu + 1$, we shall have $s(4\mu . 2m - 2\mu + 2) + s(4\mu . 2m - 2\mu + 1) = -16s\mu^2 + (16m + 12)s\mu = \frac{8}{3}m^3 + 6m^2 + \frac{10}{3}m = \frac{1}{6}(2n^3 + 3n^2 - 2n - 3)$ varieties, when n is odd.

4. The only supposition which remains is, that b, d, e, g, are all different numbers. In this case there are eight changes of position, viz. $\begin{array}{ccc} & b & \\ d & & e \\ & g & \end{array}$, $\begin{array}{ccc} & g & \\ d & & e \\ & b & \end{array}$, $\begin{array}{ccc} & b & \\ e & & d \\ & g & \end{array}$, $\begin{array}{ccc} & g & \\ e & & d \\ & b & \end{array}$, $\begin{array}{ccc} & d & \\ b & & g \\ & e & \end{array}$, $\begin{array}{ccc} & e & \\ b & & g \\ & d & \end{array}$, $\begin{array}{ccc} & d & \\ g & & b \\ & e & \end{array}$, $\begin{array}{ccc} & e & \\ g & & b \\ & d & \end{array}$.

We may consider b, g, as the extremes, and d, e, as terms equidistant from the mean, of an arithmetical series of $b-g+1$ terms; the common difference being unity. It is then manifest, that $\frac{1}{2}(b-g-1)$ pairs of numbers may be taken for d, e, when $b-g+1$ is an even number, and $\frac{1}{2}(b-g-2)$, when it is an odd number. Now in making $g=0\ldots\ldots b-3$, while b remains unaltered, these conditions take place alternately; giving $\frac{1}{2}(b-2)+\frac{1}{2}(b-2)+\frac{1}{2}(b-4)+\frac{1}{2}(b-4)+\ldots\ldots 1+1=\frac{1}{4}(b.b-2)$ choices for d, e, when b is even, and $\frac{1}{2}(b-1)+\frac{1}{2}(b-3)+\frac{1}{2}(b-3)+\ldots\ldots 1+1=\frac{1}{4}(b-1)^2$, when b is odd. Consequently the general term of the series of variations is $2b.(b-2).(n-b+1)$ in the former case, and $2(b-1)^2.(n-b+1)$ in the latter.

In the class now under consideration it must be remembered that b can never be less than 3. When, therefore, $n=2m+2$, b in the last formulæ must be alternately $=2\mu+2$ and $2\mu+1$; μ representing as before, the variable whose limits are 1 and m. We have, therefore, $8s\,(\mu.\mu+1.2m-2\mu+1)+16s\,(\mu^2.m-\mu+1)=8\times\left\{-4s\mu^3+(4m+1)s\mu^2+(2m+1)s\mu\right\}=\frac{8}{3}(m^4+4m^3+5m^2+2m)=\frac{1}{6}n^2.n^2-4$ variations, when n is even.

When $n=2m+1$, b is alternately 2μ and $2\mu+1$, and the sum of the varieties is $16s\,(\mu-1.\mu.m-\mu+1)+8s\,(\mu^2.2m-2\mu+1)=8\times\left\{-4s\mu^3+(4m+5)s\mu^2-(2m+2)s\mu\right\}=\frac{4}{3}(2m^4+4m^3+m^2-m)=\frac{1}{6}(n^2-1.n^2-3)$ variations, when n is odd.

5. The aggregate of all the varieties of arrangement now determined is $\frac{1}{6}(n^4+6n^3+14n^2+15n+6)$, or, in a more convenient expression, $\frac{1}{6}(n.n+1.n+2.n+3)+\frac{1}{2}(n+1\times n+2)$ *universally*, whether n be an even or an odd number.

Remarks. 1. I have here taken the mean of three general methods of considering this problem.

The most extensive view of the question may be thus stated:

In how many ways may persons, chosen out of N *people, be disposed in the external cells of a square of* 9 *cells, so that there may be always n persons in each side of the square, and that no two arrangements may be identical in the united regard of numbers, selection, and grouping?*

The general arrangement, as exemplified by the first marginal figure, may be contemplated as a compages of horizontal (fig. 2,) and vertical rows (fig. 3). In either class, separately considered, the exterior rows are to consist of n persons in each, and the remaining $N - 2n$ form a reserve to supply the middle row.

The solution of the question will easily be discovered by attending to the practical mode of performing these combinations of the two classes; which would be this. Arrange all the N people once on the plan of figure 2, and once on the plan of figure 3; then let each person take his place in figure 1, in that cell which corresponds to the intersection of the horizontal and vertical rows in which he stood before. By this means the conditions of figures 2 and 3, are combined without any restriction.

Fig. 1.

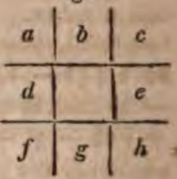


Fig. 2.

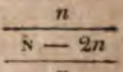
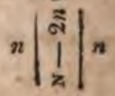


Fig. 3.



Each class admits of

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots\ldots\ldots\ldots\ldots N}{(1 \ldots\ldots n)(1 \ldots\ldots n)(1 \ldots\ldots (N - 2n))}$$

permutations; and as the permutations in each class take place exclusively in the direction of its own structure, any *one* arrangement of one class may be combined with *all* those of the other, constituting in all

$$\left\{\frac{1 \ldots\ldots\ldots\ldots\ldots\ldots N}{(1 \ldots\ldots n)^2 (1 \ldots\ldots (N - 2n))}\right\}^2 \text{ varieties.}$$

As the central cell is not mentioned in the question, those persons who happened to stand in the middle row both of the horizontal and vertical arrangements, would consider themselves at liberty to go where they pleased. Had the number and selection of these persons also been contemplated, omit the word "external" in the question, and then the total amount will be expressed by the sum of the series,

$$\frac{(1 \ldots N).(1 \ldots 2n)}{(1 \ldots\ldots n)^4}\left\{1 + \frac{2n+1}{1} + \frac{2n+1.2n+2}{(1 . 2)^2} + \ldots\ldots \frac{2n+1 \ldots\ldots N}{(1 \ldots \overline{N-2n})^2}\right\}$$

the successive terms of which are composed of the number of varieties deduced from the preceeding formula by substituting $2n$, $2n + 1$, &c. for N, and of the permutations of $2n$, $2n + 1$, &c. persons out of N.

As a comparative example in the smallest numbers possible, let $n = 1$ and $N = 4$, then on the principle I have adopted, there will be 7 varieties of disposition; but on the principle of permutations there may be 144 or even 336.

2. Proposing the question as an arithmetical puzzle, it has been usual to make the four corner numbers equal to each other, and likewise the four middle ones. With these limitations, it is readily found by simple equations that $a = n - \frac{1}{4} s$ and $b = \frac{1}{2} s - n$; consequently s cannot exceed $4n$ nor be less than $2n$, and must be divisible by four. Whence it also readily follows that the arrangement may be made in $\frac{1}{2} n + 1$ ways when n is even, or in $\frac{1}{2} (n + 1)$ when it is odd.

3. In Montucla's example, n is $= 9$, which will admit of five arrangements limited as in the last remark. Four of these were given by Ozanam. Montucla, after giving the fifth, adds also a sixth, limited only as in class 1. It is curious, that having made this step in advance, and the remarks by which it is introduced, he did not perceive that the conditions of the question admitted of very numerous solutions. In fact there are 55 of the same class as his 6; and the whole number possible is 2035.

4. Instead of a square, restricted to 3 cells in each side, we may imagine any quadrangle having in each side any nnmber of cells not less than 3, the other conditions remaining unaltered, and the solution is readily deduced from what has been done above. In fact, we have only to distribute the number b into any portions b', b'' &c. at pleasure, and dispose them in the cells between a and c, and so of the other middle numbers d, e, g.

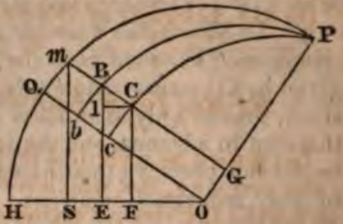
5. The principle may even be extended to polygons of any even number of sides, having 3 or more cells in each side, and so constructed that each of the angular cells shall be common to two sides. Suppose, for example, the cells of such a polygon of 2π sides are to be so filled with numbers, that while the whole sum employed varies from πn to $2 \pi n$, the sum contained in each side shall be constantly n. To effect this by means of a rule similar to that on which the preceding calculations were grounded, it is sufficient, instead of "*double the number*," to say "*π times the number*", instead of "*two parts*", "*π parts*", and instead of "*Place—opposite each other in the four middle cells*", to say "*Distribute each of these sets of π parts among half the middle cells, taken alternately.*"

ARTICLE XIII.

Investigation of a Rule for finding the Latitude.

By Mr. JOHN BRANSBY, *of Ipswich.*

At page 304, (8th edit.) of Moore's Practical Navigator, is the following rule for finding the latitude by double altitudes, "Add together the arithmetical complement of the cosine of the declination, the arithmetical complement of the common logarithm of the difference of the cosines of the times from noon, turned into degrees and minutes, and the common logarithm of the difference of the natural sines of both altitudes; the sum of these three logarithms will give the logarithmic cosine of the latitude." The same rule is in Bettesworth's Naval Mathematics.



Investigation. Let P be the north pole, QO a quadrant of the equator, and mG a quadrant of the parallel of the sun's declination. Then, the hour circles PBb, PCc, being drawn at the distances from the meridian PQH, answering to the times of the two observations, it is evident that the similar arcs bc, BC, are in the same ratio as their radii, that is, as QO : mG; but QO is the whole radius, and mG the cosine of Qm the declination: therefore BC $= m\text{G} \times bc$. Then in the right-angled triangle BIC, it will be BC $(= m\text{G} \times bc)$: BI (the difference of the sines of the altitudes) :: radius, 1 : $\left(\frac{\text{BI}}{m\text{G} \times bc} =\right)$ sin $(\angle \text{C} =)$ $\angle$QOH, the co-latitude. The rule is therefore correct. But the accuracy of the result depends on that of the times of observation; because in equal intervals the arc bc will be very different at different times of the day; and as the correct time is seldom known at sea, except in ships where the longitude is calculated from lunar observations, or where there are good chronometers; this rule appears not to be adapted to general use. One instance will shew the propriety of this remark. In Moore's third example for finding the latitude by double altitudes, the sun's declination is given $= 20^\circ$S; at $10^h\ 17^m$ per watch, its altitude was $17^\circ\ 13'$; and at $11^h\ 17^m$ it was $19^\circ\ 41'$. By this rule the latitude will be found $= 57^\circ\ 54'$ N. which is $7^\circ\ 54'$ too much. But by the common approximating rule, the first operation shews that the watch was twelve minutes too slow. If the times of observation be corrected, the result by the rule above investigated, will be $50^\circ\ 49'$; only $59'$ too much. But this is very far from being sufficiently correct. By the operation of the second approximation, it appears that the watch was 13 minutes too slow. If we repeat our work with this correction, the latitude resulting will be $49^\circ\ 58'$ which is $2'$ too little. It appears then in the present instance, that the difference of one minute in the times indicated by the watch makes $51'$ difference in the latitude;

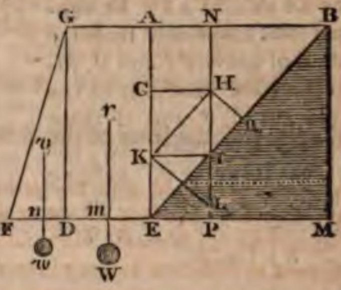
which is a farther confirmation of the defect of the rule, especially as from the rule itself it is not possible to detect the error of the watch.

ARTICLE XIV.

On the Thickness of Wharf Walls, &c. to Support a Bank of Earth.

By Mr. JOHN ADAMS, *of Stonehouse, Plymouth.*

Let ABME be a vertical section of a bank of earth, the triangular part ABE, that which is supposed to be supported by means of a wall, the vertical section of which is represented by AEFG.



Let H be the centre of gravity of the triangle ABE, through which draw HK parallel to the slope BE, and, at right angles thereto, draw KL, intersecting NHP, at right angles to AB, in L: draw HC and KI perpendicular to AE, and HQ perpendicular to BE.

From the nature of forces, if HL represent a weight proportional to the area of the triangle ABE, then HC = KI = the force perpendicular to AE, and HI = CK = the force perpendicular to the base EM. From the nature of the construction AN = EP = $\frac{1}{3}$ AB, and EK = $\frac{1}{3}$ AE.

By similar triangles we have

BE : AE :: HL : KH, BE : AB :: KH : KI, AB : AE :: KI : IH.

From whence $BE^2 : AE . AB :: HL : KI$, $BE^2 : AE^2 :: HL : IH$;

Therefore $KI = \frac{AE \times AB}{BE^2} \times HL$, and $IH = \frac{AE^2}{BE^2} \times HL$; hence

$$KI \times \tfrac{1}{3} AE \text{ or } IH \times \tfrac{1}{3} AB = \frac{AE^2 \times AB}{3 \times BE^2} \times HL.$$

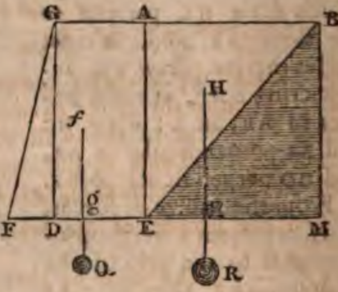
But HL represents a weight proportional to the triangle AEB or to $\frac{1}{2}$ AE × AB, therefore KI × $\frac{1}{3}$ AE is proportional to $\frac{AE^2 \times AB}{3 \times BE^2} \times \frac{1}{2} AE \times AB$; that is to $\frac{AE^3 \times AB^2}{6 \times BE^2}$ or its equal $\frac{AE^3 \times \sin^2 AEB}{6}$.

Put AE $= a$, DE $= 2z = x$, then $z = \frac{x}{2}$; FD $= 6r = \frac{a}{c}$, then $r = \frac{a}{6c}$.

We now have the area EG = DE × EA $= 2az = W$,
and the area DFG $= \frac{1}{2}$ FD × DG $= 3ar = w$,
also E$m = \frac{1}{2}$ ED $= z$, and En = ED + Dn $= 2z + 2r$.

Hence Em × W + En × $w = 2az^2 + 6arz + 6ar^2 =$ $(z^2 + 3rz + 3r^2) \times 2a = \left(x^2 + \frac{ax}{c} + \frac{a^2}{3c^2}\right) \times \frac{a}{2}$.

If m = specific gravity of the wall,
n = specific gravity of the earth,
s = sine of the angle AEB, to radius 1, and if the weights Q and R, proportional to the areas AGFE and ABE, be supposed to be suspended from the centres of gravity f and H, intersecting the base FM in the points g and P and if these weights, appended to the straight lever gP, balance each other on the fulcrum E, it is evident that Eg × Q = EP × R; that is,



$$\left(x^2 + \frac{ax}{c} + \frac{a^2}{3c^2}\right)\frac{am}{2} = \frac{a^3s^2n}{6},$$

from whence $$x^2 + \frac{ax}{c} + \frac{a^2}{3c^2} = \frac{a^2s^2n}{3m} \text{ ....(A)}$$

Therefore $$x = \left\{\sqrt{\frac{4nc^2s^2 - m}{3m}} - 1\right\}\frac{a}{2c}.$$

Cor. 1. If the section of the wall be a triangle, $x = 0$, and FD $= \frac{a}{c}$, and we have from equation (A)

$$\frac{\text{FD}^2}{3} = \frac{a^2s^2n}{3m}; \text{ and therefore FD} = as\sqrt{\frac{n}{m}}.$$

Cor. 2. If the section of the wall be rectangular, $\frac{a}{c} = 0$, and we then get from equation (A)

$x^2 = \frac{a^2s^2n}{3m}$; and therefore $x = as\sqrt{\frac{n}{3m}}$.

Cor. 3. While $c, m, n, s,$ remain the same, the breadth of the wall will vary as its perpendicular height.

Example. Given AE $= 18$ feet, AB $= 15$, $c = 8$, $m = 675$, $n = 496$, and thence $s^2 = \frac{225}{549}$; to find the thickness of the wall.

$x = \left\{\sqrt{\frac{4nc^2s^2 - m}{3m}} - 1\right\}\frac{a}{2c} = 4.54 =$ the breadth at top,

and $\frac{18}{8} + 4.54 = 6.79 =$ the breadth at bottom.

According to the principles given by Dr. Hutton, in the third volume of his Course of Mathematics, the thickness of the wall at the base $= a\sqrt{\left(\frac{ns^2}{3m} + \frac{1}{3c^2}\right)}$, which expression may be found by a *simple equation*; by which, and with the numbers given in the above example,

the thickness of the wall at bottom is 5.85 feet,
and its thickness at the top ....3.6 feet,

less by one foot than the thickness found above.

When the profile of the wall is rectangular, the thickness $\left(as\sqrt{\frac{n}{3m}}\right)$ is the same by both methods.

It will be observed that the thickness of the wall has been determined without reducing the expression $\frac{a^3s^2n}{6}$ on account of friction, if such reduction had been made, it is evident that the thickness would have been less; even as it is, it would not be trusted by practical men in the erection of Wharf Walls, &c. According to the data given in the preceding example, the thickness at the base would have been made upwards of *Seven Feet*; and from the observations that I have had an opportunity of making on works of this kind, I am of opinion that the breadth at the bottom, should be very little, or any thing less than seven feet. This, however, will principally depend on the magnitude of the stones the wall is built with, the quality of the mortar or cement used, the manner the work is executed, and the kind of earth, &c. it is backed up with.

In the failures that I have witnessed, the walls appeared first to bulge out a little below the middle, and separate a few feet from the bottom, and I have no doubt but that is generally the case. Hence, and from other considerations, it appears that the point of application K, as proposed by Dr. Hutton in his Course of Mathematics (5th Edit.) is more likely to bring out conclusions nearer the truth than the point C, heretofore chiefly used, provided the *stabilitating forces** of the masonry and supported earth could be more satisfactorily ascertained. However erroneous may be the results arising from the consideration of the action of those forces on the principle of the lever, it is perhaps the only mathematical way of obtaining *approximate* solutions. The nature of the question being such as to elude in a great measure mathematical enquiry, may be the reason of so little having been said respecting it in the best treatises on Mechanics.

March 17*th*, 1817.

ARTICLE XV.

Solution of a Dynamical Problem, by A. B.

To find the motion of a body animated by two forces one of which acts in the direction of the radius vector and the other at right angles to it.

Let AB be a portion of the curve described by a body B acted upon by a force P in the direction of the radius vector BC, and by a force P′ acting at B in a direction perpendicular to BC. Put $a = AC$, $x = AP$, $y = BP$, $z = BC$ and $v =$ the velocity of the body in the curve at B.

By the resolution of forces P will be divided into the two forces $\frac{a-x}{z}P$ and $-\frac{y}{z}P$ acting in the directions AC and BP; and P′ into the two $\frac{a-x}{z}P'$ and $\frac{y}{z}P'$ acting in the same directions: wherefore the whole accelerating force in the direction AC is $= \frac{a-x}{z}P + \frac{y}{z}P'$ and in the direction BP $= \frac{a-x}{z}P' - \frac{y}{z}P$: therefore by dynamics

$$\frac{\ddot{x}}{\dot{t}^2} = \frac{a-x}{z}P + \frac{y}{z}P'$$

$$\frac{\ddot{y}}{\dot{t}^2} = \frac{a-x}{z}P' - \frac{y}{z}P$$

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* This expression is used by Colonel Paisley in his “ Military Instructions” Vol 3, where he treats on Revetments, and in which he has given an account of a great number of ingenious experiments made with models backed with shingles.

or putting φ = the measure of the angle ACB, radius being 1, $\frac{a-x}{z} = \cos\varphi$ and $\frac{y}{z} = \sin\varphi$, whence

$$\frac{\ddot{x}}{\dot{t}^2} = \cos\varphi\ \mathrm{P} + \sin\varphi\ \mathrm{P}' \ldots\ldots\ldots\ldots 1$$

$$\frac{\ddot{y}}{\dot{t}^2} = \cos\varphi\ \mathrm{P}' - \sin\varphi\ \mathrm{P} \ldots\ldots\ldots\ldots 2.$$

Multiplying the first of these equations by $\cos\varphi$ and the second by $\sin\varphi$, and taking the difference of the products

$$\frac{\ddot{x}\cos\varphi - \ddot{y}\sin\varphi}{\dot{t}^2} = \mathrm{P}.$$

Again, multiplying the first of the above equations by $\sin\varphi$ and the second by $\cos\varphi$, and adding the results

$$\frac{\ddot{x}\sin\varphi + \ddot{v}\ \cos\varphi}{\dot{t}^2} = \mathrm{P}'.$$

But since $a - x = z\cos\varphi$ and $y = z\sin\varphi$, by taking second fluxions we get

$$\ddot{x}\ \cos\varphi - \ddot{y}\sin\varphi = z\,\dot{\varphi}^2 - \ddot{z}$$

$$\text{and}\quad \ddot{x}\sin\varphi + \ddot{y}\ \cos\varphi = 2\dot{z}\dot{\varphi} + z\,\ddot{\varphi}$$

$$\text{therefore}\qquad \mathrm{P} = \frac{z\ \dot{\varphi}^2 - \ddot{z}}{\dot{t}^2} \ldots\ldots\ldots\ldots\ldots(a)$$

$$\text{and}\qquad \mathrm{P}' = \frac{2\ \dot{z}\ \dot{\Phi} + z\ \ddot{\Phi}}{\dot{t}^2} \ldots\ldots\ldots\ldots(b).$$

Moreover if equation 1 be multiplied by $\dot{x}$ and equation 2 by $\dot{y}$ and the products added,

$$\frac{\dot{x}\ \ddot{x} + \dot{y}\ \ddot{y}}{\dot{t}^2} = (\dot{x}\ \cos\varphi - \dot{y}\ \sin\varphi)\ \mathrm{P} + (\dot{x}\sin\varphi + \dot{y}\cos\varphi)\ \mathrm{P}',$$

or substituting for $\dot{x}$ and $\dot{y}$ their equals $-(z\ \cos\varphi)^{\cdot}$ and $(z\sin\varphi)^{\cdot}$,

$$\frac{\dot{x}\ \ddot{x} + \dot{y}\ \ddot{y}}{\dot{t}^2} = \mathrm{P}'\ \dot{\varphi} - \mathrm{P}\ \dot{z}.$$

Taking the fluents

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{t}^2} = c + 2\int(\mathrm{P}'\ \dot{\varphi} - \mathrm{P}\dot{z});$$

$$\text{therefore}\qquad \mathrm{V} = \sqrt{(c + 2\int(\mathrm{P}'\ \dot{\varphi} - \mathrm{P}\ \dot{z}))} \ldots\ldots(c)$$

and the three formulæ *a*, *b*, *c* are sufficient for determining all the circumstances of the motion in any proposed case.

If v be put for the velocity in the direction BC, and u for the angular velocity at the distance r from the centre, then $v = \frac{\dot{z}}{\dot{t}}$ and $u = \frac{r\dot{\varphi}}{\dot{t}}$; therefore by substituting in formulae a and b, we have

$$P = \frac{z\,u^2}{r^2} - \frac{v\,\dot{v}}{\dot{z}}$$

$$P' = \frac{2\,u\,v}{r} + \frac{z\,v\,\dot{u}}{r\,\dot{z}};$$

the same expressions that Landen has found at the beginning of his 7th Memoir by a very different method.

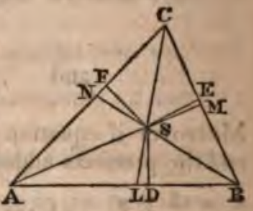
ARTICLE XVI.

An Indeterminate Problem. By *Mr.* CUNLIFFE, *R. M. C.*

PROBLEM.

To find a plane triangle such that its sides, perpendiculars, and lines bisecting the angles may all be expressed by rational numbers.

SOLUTION.



Let ACB be a plane triangle, CL, AM and BN lines bisecting the angles and terminating in the opposite sides, and intersecting in S, the centre of the inscribed circle (Euc. 4. 4.) Draw SD, SF and SE perpendicular to the sides AB, AC and BC; then will SD = SF = SE; CE = CF, BE = BD and AD = AF.

Put $CE = CF = m^2 - n^2$, and $SE = SF = 2mn$;
$AF = AD = r^2 - s^2$, and $SD = SF = 2rs$;
$BE = BD = p^2 - q^2$, and $SE = SD = 2pq$:

then $CS^2 = CE^2 + SE^2 = (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$
$AS^2 = AD^2 + DS^2 = (r^2 - s^2)^2 + (2rs)^2 = (r^2 + s^2)^2$
$BS^2 = BE^2 + SE^2 = (p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$:

whence $CS = m^2 + n^2$, $AS = r^2 + s^2$, $BS = p^2 + q^2$;

and hence we get

$AB = AD + DB = r^2 - s^2 + p^2 - q^2$,
$AC = AF + FC = r^2 - s^2 + m^2 - n^2$,
$BC = BE + EC = p^2 - q^2 + m^2 - n^2$,

and $2mn = 2rs = 2pq$, or $mn = rs = pq$.

Now $12 \times 1 = 6 \times 2 = 3 \times 4$, from whence it is evident that we may take
$m = 12, n = 1; r = 6, s = 2; p = 4$ and $q = 3$, and hence $AB = 32 + 7 = 39$; $AC = 32 + 143 = 175$; $BC = 7 + 143 = 150$.

It will be easy to shew that the sides, perpendiculars and lines bisecting the angles of triangles, determined in the foregoing manner will be expressed by rational numbers.

In the first place to show that the perpendiculars will be expressed by rational numbers.

It is well known and pretty obvious that $\frac{AB + AC + BC}{AB} \times SD =$ the perpendicular from C upon AB, which must be rational, because AB, AC, BC, and SD are all rational; and in the same manner we may shew that the other two perpendiculars must be rational.

Secondly. To show that the lines bisecting the angles must be rational.

CL bisects the angle ACB; therefore $AC : BC :: AL : LB$, and by composition, $AC + BC : BC :: AL + LB = AB : LB$, and by alternation, $AC + BC : AB :: BC : LB$. Also the line BS bisects the angle CBL (CBA); therefore $BC : LB :: CS : SL$; wherefore $AC + CB : AB :: CS : SL = \frac{AB \times CS}{AC + BC}$.

Now AB, AC, BC, and CS, are all rational, therefore SL, and consequently CL will be rational. And in the same manner it may be proved that the other two lines AM and BN will be rational.

The length of the perpendicular from C upon AB will be $\frac{364 \times 24}{39} = \frac{364 \times 8}{13}$. The length of the perpendicular from B upon AC will be $\frac{364 \times 24}{175} = \frac{52 \times 24}{25}$. And the length of the perpendicular from A upon BC will be $\frac{364 \times 24}{150} = \frac{364 \times 4}{25}$.

Again from what has been deduced, $CL = CS + SL = CS + \frac{AB \times CS}{AC + BC} = \frac{AB + AC + BC}{AC + BC} \times CS = \frac{364 \times 145}{325} = \frac{28 \times 29}{5}$.

And in the same manner we get $AM = \frac{AB + AC + BC}{AC + AB} \times AS =$

$\frac{364 \times 40}{214} = \frac{364 \times 20}{107}$; and $\text{BN} = \frac{\text{AB} + \text{AC} + \text{BC}}{\text{AB} + \text{BC}} \times \text{BS} = \frac{364 \times 25}{189} = \frac{52 \times 25}{27}$.

We shall now add another example of finding the sides of a triangle, whose perpendiculars, and lines bisecting the angles are all rational numbers.

$12 \times 2 = 8 \times 3 = 6 \times 4$, wherefore we may take $m = 12$, $n = 2$; $r = 8$, $s = 3$; $p = 6$ and $q = 4$. And from what has been done $\text{AB} = 55 + 20 = 75$; $\text{AC} = 55 + 140 = 195$; $\text{BC} = 20 + 140 = 160$; $\text{SD} = \text{SE} = \text{SF} = 2mn = 48$; $\text{CS} = m^2 + n^2 = 148$; $\text{AS} = r^2 + s^2 = 73$; $\text{BS} = p^2 + q^2 = 52$.

The length of the perpendicular from C upon AB is $\frac{430 \times 48}{75} = \frac{86 \times 16}{5}$.

The length of the perpendicular from B upon AC is $\frac{430 \times 48}{195} = \frac{86 \times 16}{13}$.

And the length of the perpendicular from A upon BC is $\frac{430 \times 48}{160} = 43 \times 3 = 129$.

Moreover $\text{CL} = \frac{430 \times 148}{355} = \frac{86 \times 148}{71}$;

$$\text{AM} = \frac{430 \times 73}{270} = \frac{43 \times 73}{27};$$

$$\text{BN} = \frac{430 \times 52}{235} = \frac{86 \times 52}{47}.$$

When the sides are rational, the segments of the sides made by perpendiculars from the opposite angles will be rational.

For it is well known, that the difference of the squares of any two sides is equal to the rectangle under the third side and the difference of the segments thereof, made by a perpendicular from the opposite angle; therefore, when the sides are rational, the difference of the segments of the sides will be rational, and consequently the segments themselves will be rational.

Also when the sides of a triangle are rational, the segments of the sides made by lines bisecting the opposite angles will be rational.

For $\text{AC} + \text{BC} : \text{AB} :: \text{AC} : \text{AL} = \frac{\text{AB} \times \text{AC}}{\text{AC} + \text{BC}}$ and $\text{AC} + \text{BC} : \text{AB} :: \text{BC} : \text{BL} = \frac{\text{AB} \times \text{BC}}{\text{AC} + \text{BC}}$.

Which expressions are both rational, because AB, AC and BC are all rational. And the segments AM, MC; and BN, NC, may in the same manner be proved to be rational.

Therefore, we have completely shown, the method of finding numerical values for the sides of a plane triangle, whose perpendiculars, lines bisecting its angles, and segments of the sides made by the said perpendiculars and lines bisecting the angles, will all be expressed by rational numbers.

ARTICLE XVII.

Mathematical Scraps. By Mr. THOMAS WHITE.

I. DYNAMICS.

The principles of Dynamics, that is, the elements of the doctrine of powers, do not require the aid of geometry for their establishment. Of power or force we know nothing, except that, by it the state of a body is altered as to either motion or rest: And this alteration in the state of a body, that is the effect of the power, as to the change of place, and generally, of its motion during the lapse of the time employed, is capable of mathematical estimation. Of two powers, the energies may have either a hypothetical or other relation to each other; and this relation is the same as that of their effects. The ratio of the energies of two powers may be denoted by the ratio of the abstract numbers 1 and v'; and the ratio of the effects of these energies by the ratio of the numbers g' and $g'v' = v$, suppose, where g' is $=$ a number of feet (to be afterwards assigned) passed by the body in a second of time, and is the measure of the effect of the energy of the first power, as v is the measure of that of the second power in feet: the body being, in both cases, the same. By restricting the word power to mean that agency, whatever it is, which changes the state of the unit of the mass as to either motion or rest.—And the word force to either the abstract number which marks the energy, or, to that which measures the effect, in feet, of this energy acting on the unit of the mass, it is presumed that, clearer notions will be acquired; and, indeed, the use of even the word force is wholly unnecessary in treating dynamics. Thus, then, the energy of the power acting on the unit of the mass is denoted by an abstract number; and, its effect by a number denominated feet passed by the unit in a second of time: The space passed by the unit and the time employed present themselves naturally to the mind while contemplating the effect of the energy of power.

The following way of arranging the subject is unusual; but it is presumed to be more suited to our first notions than the previous consideration of restricted and regular agency, which results from a more mature examination of the subject.

In general, then, the unit of the mass may be urged over the space of s feet, during the time of t seconds, by an incessantly varying energy: This admits of two modifications, or cases.

CASE I. The energy of the power may be conceived to be altogether suspended instantly at the end of the time t: In this case, the space, in feet, afterwards passed by the unit during the second interval of time t will be uniformly described (Law I.); and, any part of this space which is passed in a second of time is called the velocity v of the unit at that point of s where the energy became suspended.

a. Hence, $1'' : v :: t'' : v \cdot \frac{t''}{1''} = vt =$ the space thus uniformly passed in t seconds with the acquired velocity v. This is the law of uniform motion; and, applies to the motion generated by one body impinging on another: For, time elapses and velocity is acquired during their contact.

CASE II. At the same point of s, the incessantly varying energy may be conceived instantly to cease to be variable, that is, may become constant. Then, the velocity, in feet, uniformly generated, by the supposed constant energy, during the uniform lapse of the second of time next succeeding t'', that is, during the lapse of $1''$ from o, is the *natural* measure of the intensity of the incessantly varying energy of the power acting on the unit of the mass at that point of its path s; and, may be called (for reasons soon to appear) the *radical velocity*, and be denoted by o feet.

a. Hence, during the element $\dot{t}$ of the time, which is supposed to elapse uniformly, the element $\dot{s}$ of the path will be uniformly passed by the unit with its acquired velocity v: Hence

we have . . $1'' : v :: \dot{t} : \dot{s}$; and, therefore, $v = \frac{\dot{s}}{\dot{t}}$; and $\dot{v} = \frac{\ddot{s}}{\dot{t}}$, the increase of velocity in $\dot{t}''$;

and, by II, . . . $1'' : o :: \dot{t} : \dot{v} = o \cdot \frac{\dot{t}}{1''} = o\dot{t}$ the increase (from the root o) of velocity in $\dot{t}''$; hence $o = \dot{v}\,\frac{1''}{\dot{t}'} = \ddot{s} \cdot \frac{1''}{\dot{t}''} \cdot \frac{1''}{\dot{t}''}$, where the homogenity of the terms is visible; that is, o is $=$ to either

$\frac{\dot{v}}{\dot{t}}$, where v is concerned, or to $\frac{\ddot{s}}{\dot{t}^2}$, where v is unnecessary; and is the most general measure, in feet, as s is expressed in feet, of the incessantly varying energy of a power acting on the unit over the space s during the time t. Both forms require the integral calculus. It must be kept in mind that, of the symbols o' and o, the former is what is usually called the "*accelerative force*"; and the latter is the absolute measure of it in feet; *i. e.* of the same denomination as s.

b. If at the end of the first interval t'', the incessantly varying energy of the power become constant, (as in I. *a.*), and continue so during the lapse of the next t'': Then, the velocity, uniformly generated from o as a root, increasing as the time t uniformly increases, will, in this 2nd portion of time, be ot.; for, as before, $1'' : o :: t : ot$; call this acquired velocity v'; and the space thus passed in this 2nd portion of time, call s': Then ....$1''$ $: v' :: \dot{t} : \dot{s}'$; and $v' = \frac{\dot{s}'}{\dot{t}}$; also..$1'' : o :: t : v' = ot$; and theref. $\dot{s}' = ot\dot{t}$; and $s' = o . \int t\dot{t} = o . \frac{t^2}{2}$, when the unit proceeds from rest. Again, if we put s'' for the space uniformly passed by the unit, with the uniformly acquired velocity v', in the 3rd portion of time t, we have, as before, $ot = v'$; and, (by I. *a.*) $1'' : v' :: t : s'' = v't$, the space uniformly passed in t'' with the velocity v': Hence, $s'' = tv' = t^2 o$: Now we have found $s' = o . \frac{t^2}{2}$, hence $s'' = 2s'$; that is, the space s'' uniformly passed with the velocity v' during the time t, in which v' has been uniformly acquired from rest, is *double* of s', the space passed in acquiring v' in the same time t.

c. Hence, the radical velocity o is $= \frac{2s'}{t^2}$ (without involving the vel.) $= \frac{v'}{t}$ (without involving the space) $= \frac{s''}{t^2}$; and if $t = 1''$, we have $o = 2s' =$ double the space passed in $1''$, in which o is acquired from rest.

d. Hence, also for any time T, in which the velocity V is uniformly acquired from the same root o, the unit passing uniformly over the space S with the velocity V, we have $\mathrm{S} = o\mathrm{T}^2$: theref. (II. *b*) $s'' : \mathrm{S} :: v't : \mathrm{VT} :: ot^2 : o\mathrm{T}^2 :: t^2 : \mathrm{T}^2 :: v'^2 : \mathrm{V}^2$; for, $t : \mathrm{T} :: v' : \mathrm{V}$. These results include all that belongs to what is called *uniform acceleration.*

e. These results are yet to be adapted to the known measure, in feet, of the energy of some natural power. By experiments, the measure of the energy of the power called gravity has been discovered. For this energy is found to be constant near the earth; and such as to urge any mass whatever over $16\frac{1}{12}$ feet in the first second of time, in our latitude. Hence, of two homogeneous bodies, the energy of the power called gravity on the one is to that of gravity on the other as either their weights or their masses.

f. At the ends of a flexible thread hanging over a fixed pulley (neglecting friction, &c.) conceive the homogeneous masses A and B the less, to be attached: Then, the energy of the power producing motion in the system is, in this case, denoted by the weight (II. *e.*) of the mass A — B; and the energy of the power opposed to the power producing motion, by the weight of the mass A + B;

Hence . . A + B, the weight moved, usually called the mass moved which may be designed by (ms);

: A — B, the moving weight, called usually the moving force, and may be designed by (mf);

:: 1, an abstract number, denoting the weight of the unit of the mass;

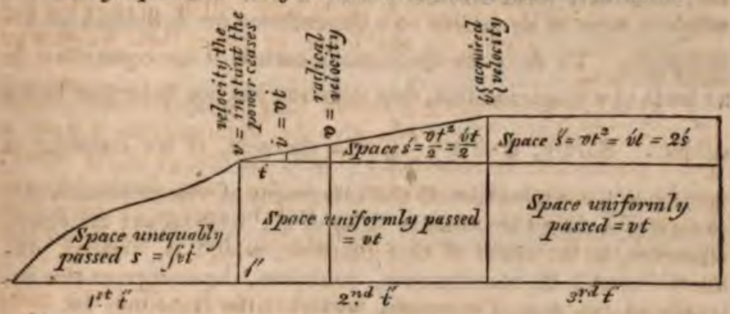
: $\frac{A - B}{A + B} = \frac{(mf)}{(ms)}$, an abstract number, denoting the restricted energy of the power acting on the unit $= \upsilon'$; and is, in this case, invariable; for A and B are known. Hence, if (mf) be $= (ms)$, then $\upsilon' = 1$, an abstract number denoting the unrestricted energy of the power of gravity on the unit; consequently, if we denote $16\frac{1}{12}$ feet by g, the velocity generated in $1''$ is (by II. *b.*) $= 2g$; consequently 1 (energy of grav.) : $2g$ (its effect in feet) :: $\upsilon' = \frac{(mf)}{(ms)}$ (the restricted energy) : $2g \cdot \frac{(mf)}{(ms)} = 2g \cdot \upsilon'$ (its measure in feet) $= \upsilon$: Hence, g' (used at the commencement) is $= 2g = 32\frac{1}{6}$ feet; and, hence, the absolute velocity acquired in t'', by a constant power, is (II. *b.*) $= \frac{t''}{1''} \cdot \upsilon = 2g \cdot t'' \cdot \upsilon'$; where υ' is an abstract number called usually the *accelerative force.*

If $\frac{(mf)}{(ms)} = \upsilon'$ be expressed by a function of variable quantities, we have 1 (energy of gravity) : $2g$ (its effect in feet) :: $\upsilon' = \frac{(mf)}{(ms)}$ (the varying energy of the power) : $2g \cdot \upsilon' = \upsilon$ its effect in feet. *Here* it must be remembered that υ' is an abstract number; and that, in this case,

$2g . \mathfrak{v}' = \mathfrak{v}$ is $= \frac{\dot{v}}{\dot{t}}$, when a function expressing v is known

and $2g . \mathfrak{v}' = \mathfrak{v}$ is $\frac{\ddot{s}}{\dot{t}^2}$, without the expression of v.

A Synoptical view of the above Method and the Results.


v = velocity the instant the power ceases
$\dot{v} = \mathfrak{v}\dot{t}$
$\mathfrak{v}$ = radical velocity
$\dot{v}$ acquired velocity
Space $\dot{s} = \frac{\mathfrak{v}t^2}{2} = \frac{\dot{v}t}{2}$
Space $\ddot{s} = \mathfrak{v}t^2 = \dot{v}t = 2\dot{s}$
$\dot{t}$
Space unequably passed $s = \int v\dot{t}$
Space uniformly passed $= vt$
Space uniformly passed $= vt$
t''
1st t″
2nd t″
3rd t″

The space s, described with an incessantly varying motion, is represented by a curvilinear area: because, the abscissa represents the time elapsing uniformly; and the ordinate, at any point of time, represents the velocity v at that point of time; and $\dot{s}$, being $= v\dot{t}$, s is $= \int v\dot{t}$. The space s', passed with an uniformly accelerated motion is represented by a right-angled triangular space; because $s' = \frac{1}{2}$ base $\times \perp$; and, the space s'' is represented by a rectangle; for $s'' = v' \times t = \perp \times$ base $= 2s'$.

SCHOLIUM 1°. If v be the velocity of the unit of the mass at the end of the time t'' in which it has passed the space s; and $\mathfrak{v}'$, an abstract number, denote the energy of the power, usually called the *accelerative force,* at that point of s, and, which is measured by $\mathfrak{v}$ feet passed in 1″, as 1 (unity) denotes the energy of gravity which is measured by $2g = 32\frac{1}{6}$ feet passed in 1″: Then,

1°. If $\mathfrak{v} = 2g . \mathfrak{v}' = 0$; s is passed uniformly with the velocity acquired, and is $= vt$.

2°. If $\mathfrak{v} = 2g . \mathfrak{v}'$ be constant, v is uniformly acquired and is $= t\mathfrak{v} = 2g . t . \mathfrak{v}'$: Also, s, passed with an uniformly accelerated motion, is $= \frac{1}{2}\mathfrak{v}t^2 = 2g . \mathfrak{v}' . \frac{t^2}{2} = g . t^2 . \mathfrak{v}'$.

3°. If $\mathfrak{v} = 2g . \mathfrak{v}'$ be variable, $\dot{v} = \mathfrak{v}\dot{t} = 2g . \mathfrak{v}' . \dot{t}$: And, $\dot{s} = v\dot{t}$, when v is expressed: Also, in this case, $\ddot{s} = \mathfrak{v} . \dot{t}^2 = 2g . \mathfrak{v}' . \dot{t}^2$; which is independent of v.

SCHOLIUM 2°. Various opinions have been entertained of the equation $\dot{v} = \mathfrak{d} . \frac{\dot{t}''}{1''}$, *i. e.* $\dot{v} = \mathfrak{d}\dot{t}$, which is admitted by all to be fundamental. D'Alèmbert, Bezout, &c. consider it to be a definition or an assumption and are followed, in this, by some celebrated names in this country: The reasons given appear, to me, singularly unsatisfactory; and, if well founded, reduce the whole system of dynamics to a dependence on definition or assumption. To deny the theorematic nature of the equation $\dot{v} = \mathfrak{d}\dot{t}$ leads to a suspicion that, due attention has not been paid to the subject. Surely, $\dot{v} = \mathfrak{d} . \frac{\dot{t}''}{1''}$ is a theorem, if we consider it merely as the expression of the increment of velocity uniformly generated during the uniform lapse of 1″, without any attention, whatever, to the cause of this increase, which is, indeed, unknown to us. By this theorem, doubtless, the energy or the intensity of the cause is measured, whatever the cause may be.

To rest on a definition in physics, is certainly not so safe as to do so in matters purely geometrical. The above equation shews the relation between the measure $\mathfrak{d}$ (in feet, for instance) of the energy of the power;—the element $\dot{v}$ of the velocity generated;—and, the element $\dot{t}$ of the time employed: Of these nothing seems to be obscure; and, the equation itself is bottomed on the 2nd case or modification of the most general view of incessantly varying energy, without the consideration of power at all. If the modification, on which the *radical velocity* or measure in feet of the energy of a power is rested, be called a definition, because it is *assumed* as the measure of the energy by which the unit is urged at that point of its path—and, here their argument should have begun—it is an assumption which is so consonant to the laws by which our judgment is regulated that, it possesses the simplicity and character of an axiom; and, admits of explanation but not of proof.

II. CONICS.

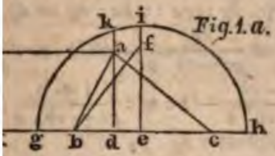
To derive the properties of the ellipse, the hyperbola, and the parabola, by means of the circle, independent of either their "determining ratio," or the "cone."

PROP. I. The base *bc* of the triangle *cab* (fig. 1. *a*) is bisected in *e* and both ways produced to *g* and *h*, so that, $ca + ab = 2ge$

PROP. I. The base *bc* of the triangle *cab* (fig. 1. *b*.) is bisected in *e*, and the points *g* and *h* are taken therein, so that, $ca - ab =$

hen, dropping the perpendicular

$ed : ca - ab :: ge : 2eb.$



$b)\,(ca - ab) = (cd + db)\,(cd - bd)$
$ge\,.\,(ca - ab) = 2be\,.\,2ed;$
$l : ca - ab :: ge : 2eb.$

II. If about b as a centre, and dis-
$= eh$, a perpendicular at e be cut in
be drawn, and the sine and cosine
le abd be called respectively ϕ' and ϕ;

$ab = \frac{ef^2}{ge - eb\,.\,\phi}.$
$ed : ca - ab :: ge : 2eb,$
$ed : ge - ab :: ge : eb,$
$l = be - ab\,.\,\phi : ge - ab :: ge : eb;$
$.\,(ge - eb\,.\,\phi) = ge^2 - eb^2 = ef^2;$
uently $ab = \frac{ef^2}{ge - eb\,.\,\phi}.$

(fig. 1, a). On gh describe a semi-
and produce da and ef to cut it in

ecause $ef^2 = ge^2 - eb^2$; and $bd =$
$\overline{\phi}$; and $ad = \frac{ef^2\,.\,\phi'}{ge - eb\,.\,\phi}$;

$-eb) + \frac{ef^2\,.\,\phi}{ge - eb\,.\,\phi} = (ge - eb)\,.\,eg\,.\,\frac{1 + \phi}{ge - eb\,.\,\phi}$

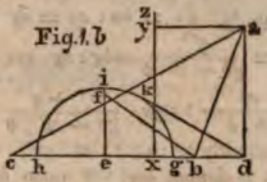
$eb) - \frac{ef^2\,.\,\phi}{ge - eb\,.\,\phi} = (ge + eb)\,.\,eg\,.\,\frac{1 - \phi}{ge - eb\,.\,\phi}:$

$dh = (ge^2 - eb^2)\,.\,eg^2\,.\,\frac{1 - \phi^2}{(ge - eb\,.\,\phi)^2} =$
$\frac{\phi'^2}{(ge - eb\,.\,\phi)^2}$, that is $dk = ef\,.\,eg$
$\overline{.\,\phi)}$; which is to $da = \frac{ef^2\,.\,\phi'}{ge - eb\,.\,\phi}$,
; that is, as ei to ef, a given ratio:
also, dk is > than da.

. Hence $ad^2 : ef^2 :: dk^2 : ei^2$; that
$:: gd\,.\,dh : ge^2;$
$^2 = (ge + eb)\,(ge - eb) = bh\,.\,bg$
of a ⊥ from b to the circle.

$2ge = 2eh$; then, dropping the perpendicular ad;

$$ed : ca + ab :: ge : 2eb.$$



For, $(ca + ab)\,(ca - ab) = (cd + db)\,(cd - db)$,
that is $(ca + ab)\,.\,2ge = 2de\,.\,2eb$;
hence $ed : ca + ab :: ge : 2eb.$

Prop. II. If about g, as a centre, and distance $be = ec$, a perpendicular at e be cut in f, and bf be drawn; and the sine and cosine of the angle abd be called respectively ϕ' and ϕ,

then, $ab = \frac{ef^2}{ge - eb\,.\,\phi}.$

By I. $ed : ca + ab :: ge : 2eb,$
that is $ed : ge + ab :: ge : eb,$
or, $eb + bd = eb + ab\,.\,\phi : ge + ab :: ge : eb;$
theref. $ab\,.\,(ge - eb\,.\,\phi) = eb^2 - eg^2 = ef^2;$
Consequently $ab = \frac{ef^2}{ge - eb\,.\,\phi}.$

Cor. 1. (fig. 1, b). On gh describe a semicircle gih; and draw the tangent dk. Then because $ef^2 = be^2 - eg^2$;

and $bd = \frac{ef^2\,.\,\phi}{ge - eb\,.\,\phi}$; and $ad = \frac{ef^2\,.\,\phi'}{ge - eb\,.\,\phi}$;

we have

$$gd = (be - eg) + \frac{ef^2\,.\,\phi}{ge - eb\,.\,\phi} = (be - eg)\,.\,ge\,.\,\frac{1 + \phi}{ge - eb\,.\,\phi}$$

and,

$$dh = (be + eg) + \frac{ef^2\,.\,\phi}{ge - eb\,.\,\phi} = (be + eg)\,.\,ge\,.\,\frac{1 - \phi}{ge - eb\,.\,\phi}:$$

and hence

$$dk^2 = hd\,.\,dg = eg^2\,.\,(be^2 - eg^2)\,.\,\frac{1 - \phi^2}{(ge - eb\,.\,\phi)^2};$$

that is, $dk = eg\,.\,ef\,.\,\frac{\phi'}{ge - eb\,.\,\phi}$; which is to

$da = \frac{ef^2\,.\,\phi'}{ge - eb\,.\,\phi'}$, as ge to ef;

that is, as ei to ef, a given ratio: Hence ef may be either >, =, or < than ei.

Cor. 2. Hence $ad^2 : ef^2 :: dk^2 = gd\,.\,dh : ge^2$: Hence, also, $ef^2 = (be + eg)\,(be - eg) = hb\,.\,bg =$ to the square of a tangent from b the circle.

Prop. III. A point x may be found in hg produced, such that, drawing a perpendicular xz to gh, and ay parallel to it,

$$bg : gx :: ba : ay.$$

By hyp. $bg : gx :: ba : ay = dg + gx ::$

$$ba - bg = \frac{ef^2}{ge - eb \,.\, \phi} - bg : bg \,.\, ge \,.\, \frac{1+\phi}{ge - eb \,.\, \phi}$$

$$:: \frac{ge + eb}{ge - eb \,.\, \phi} - 1 : ge \,.\, \frac{1+\phi}{ge - eb \,.\, \phi}$$

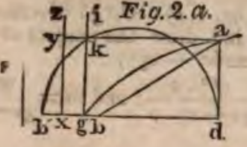
Consequently $bg : gx :: eb : ge$; hence gx is found, and the ratio bg to gx is that of *minor* to *major*, and is called the *determining ratio.*

Schol. The locus of a is called an ellipsis of which b and c are the foci; ge and ef the semiaxes and xz the directrix: Hence all the properties of the curve may be found.

Prop. IV. Produce bg both ways; take $gb' = 4gb$; and in it, taking any point d, describe a semicircle on $b'd$ cutting the perpendicular gki in k, and draw ka parallel to gd, and da perpendicular to it; meeting ka in a; then it is plain that, $da^2 = gk^2 = b'g \,.\, gd = 4bg \,.\, gd$; therefore, the point a is in a parabola of which g is the vertex, and b the focus.

The above is a simple method of describing a parabola.

Fig. 2. a.



Prop. III. A point x may be found in such that, drawing xy perpendicular to gh, ay parallel to it,

$$bg : gx :: ba : ay.$$

By hyp. $bg : gx :: ba : ay = dg + gx ::$

$$ab - bg = \frac{ef^2}{ge - eb \,.\, \phi} - bg : bg \,.\, ge \,.\, \frac{1+}{ge - eb.}$$

$$:: \frac{ge - eb}{ge - eb \,.\, \phi} - 1 : eg \,.\, \frac{1+\phi}{ge - eb \,.\, \phi};$$

Consequently $bg : gx :: be : eg$;

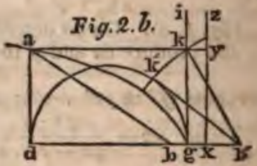
hence gx is found, and the ratio of $bg : gx$ that of *major* to *minor*; and is called the *termining ratio.*

Schol. The locus of a is called a hyperb of which b and c are the foci; ge and ef semiaxes and xz the directrix.

And, hence all the properties may be found

Prop. IV. Produce bg both ways; $gb' = 4gb$; and in it, taking any point d, scribe a semicircle on gd; draw a tangent with which cut, in k, a perpendicular draw ka parallel to $b'd$ and da perpendicul to it, meeting in a: Then, a is in a parabol For, $gb' \,.\, b'd = b'k'^2 = bk^2$; that is $4bg \,.\,$ ($+ gd) = b'k'^2 = gk^2 + (4gb)^2$; that is $gd = gk^2 = ad^2$; and g is the vertex and b focus.

Fig. 2. b.



In both figures (2. *a.* and 2. *b.*) draw ba; and put the sine and cosine of angle $abd = \phi'$ and ϕ; then, $ad = ab \,.\, \phi$; and $bd = ab \,.\, \phi$; and because $da^2 = 4b$ $(gb + bd)$; we find $ab = \frac{2bg}{1-\phi}$: Also, a point x may be found such, that $gx :: ba : ay = gd + gx$; and, this ratio will be found that of *equality*; a the perpendicular from x is the directrix.

By conceiving the points g, b and a (fig. 1. a; 1. b.) to remain fixed in ellipse and hyperbola, while ge is indefinitely lengthened, the arc gk will co tinually tend to, and will ultimately coincide with a perpendicular gd at g; a ka will continually tend to be and will ultimately become parallel to gd; and w intersect the perpendicular da in a, a point in the *new* curve: And, as the ellip and hyperbola require each a fixed circle, so their parabolas require each anal gously, each a variable circle; the one $b'kd$ originating at b'; and, the other originating in g; and, the hyperbola and its parabola require each a fixed exteri point c and b'.

If we conceive another ordinate $a'd'$ drawn to the ellipse, we have (II. cor. $ad^2 : a'd'^2 :: gd \,.\, dh : gd' \,.\, d'h$: Now let h be indefinitely removed, dh will ul nately be $= d'h$; and, hence $ad^2 : a'd'^2 :: gd : gd' :: 4bg \,.\, gd : 4bg \,.\, gd'$; an ence, if $ad^2 = 4bg \,.\, gd$; then $a'd'^2 = 4bg \,.\, gd'$, a parabola: And, thus for t *perbola.*

Mathematical School of Dumfries, March 5, 1816.

ARTICLE XVIII.

To the Editor of the Mathematical Repository.

SIR,

My letter, containing the result of an improved method of treating the problem of the BLIND ABBESS, reached you, I presume, too late for the use of your 15th Number; you will therefore oblige me by allowing a place in your next, to the following remarks, which are supposed to follow immediately after the *Rule*.

CALCULATION. Every thing hinges on the number we have denominated b. The limits of this are 0 and n inclusively. Those of a are 0 and $n-b$; admitting $n-b+1$ choices, while b remains unaltered. Those of d are b and g, which admit $b-g+1$ choices, while b and g remain constant; or $s(b-g+1)$ choices, while b remains constant; g varying from 0 to b.

Speaking generally, b may occupy either of 4 different situations at option, viz. either of the middle cells. So that for each value of b, we may now infer the number of solutions to be equal to

4 × *choices of a × sum of choices of d.*

But in the $b-g+1$ assumed choices of d, are included the cases $d=b$ and $d=g$ $\therefore e=b$; and by each of these it is obvious that exactly half of the 4 situations claimed for b, are preoccupied; so that these *two cases* are, in calculating, just equivalent to *one*. We must consequently estimate the choices of d at $b-g$ only.

And as by this arrangement we exclude the class of solutions in which $b=g$, and $\therefore d=e=b=g$, which includes one solution appertaining to each value of b and a, we must account for this class separately.

The correct number of solutions, corresponding to each value of b, or the *Terminus Generalis* of the whole series of solutions, is therefore

$$\left\{4s(b-g)+1\right\} \times \left\{n-b+1\right\}$$

the correct value of which, when g varies from 0 to b, is

$$(2b^2+2b+1) \cdot (n-b+1).$$

Hence the whole number of solutions is

$$s\left\{(2b^2+2b+1) \cdot (n-b+1)\right\}$$

whose correct value, when $b = 0 \ldots\ldots n$, is the same as was determined in my paper, but may still more neatly be expressed in the form

$$\tfrac{1}{6}\left\{(n+1 \,.\, n+2)^2 + (n+1 \,.\, n+2)\right\}.$$

I am, &c.

W. G. HORNER,

* * *

ARTICLE XIX.

To the Editor of the Mathematical Repository.

SIR,

Baron Maseres, in the second volume of his Scriptores Logarithmici, has reprinted Euclid Spiedell's Logarithmotechnia, but, by comparing his text with the original, I find that there is an omission. The pamphlet was printed in small quarto, and there was annexed to it a folio leaf which folded up.—This, most probably, had been torn off from the copy, which the Baron printed from, as it is not to be found in his book; and I therefore send you a copy of it. He is too liberal a man to suspect that the insertion of it in your Repository can be intended as any reproach to him. In a great work like his, it was impossible to avoid such an oversight, and I do not know how it can be so well repaired as by reprinting the leaf in your valuable miscellany.

Dec. 9, 1818. S.

	The Infinite Series of Numbers Proportional.			The Quotes to be Added.	
A	50,0000000000000	I	*	50,0000000000000	A
B	25	II	—	125	B
C	125	III	*	41666666666666	C
D	625	IV	—	15625	D
E	3125	V	*	625	E
F	15625	VI	—	26041666666666	F
G	78125	VII	*	11160714285 71	G
H	390625	VIII	—	48828125	H
I	1953125	IX	*	21701388888 8	I
K	9765625	X	—	9765625	K
L	48828125	XI	*	44389204545	L
M	244140625	XII	—	20345052083	M
N	1,220703125	XIII	*	9390014037	N
O	6103515625	XIV	—	4359654017	O
P	30517578125	XV	*	2034505206	P
Q	15258789062	XVI	—	953674316	Q
R	7629394531	XVII	*	448787914	R
S	3814697265	XVIII	—	211927625	S
T	1907348632	XIX	*	100386770	+
V	953674316	XX	—	47683715	
W	476837158	XXI	*	22706531	
X	238418579	XXII	—	10837208	
Y	119209289	XXIII	*	5183013	
Z	59604644	XXIV	—	2483527	
	29802322	XXV.	*	1192093	
	14901161	XXVI	—	573122	
	7450580	XXVII	*	283355	
	3725290	XXVIII	—	133046	
	1862645	XXIX	*	64229	
	931322	XXX	—	31044	
	465661	XXXI	*	15021	
	232830	XXXII	—	7276	
	116415	XXXIII	*	3527	
	58207	XXXIV	—	1712	
	29103	XXXV	*	831	
	14551	XXXVI	—	404	
	7275	XXXVII	*	196	
	3637	XXXVIII	—	96	
	1818	XXXIX	*	46	
	909	XL	—	22	
	454	XLI	*	11	
	277	XLII	—	5	
			*	2	
			—	1	

10,00000000 = The whole Sum A — B : A :: A =	Logarithm of 2	693147180559945
6,66666666 = A — C : A :: A = Impares	Half the Logarithm of 3	549306144334055
3·33333333 = Pares B — D : B :: B =	Logarithm of 3	1098612288668110
A—B . C—D . E—F . G—H = 3·333333333 A — B + C — D + E — F + = 3·333333333	Logarithm of the differ. between 2 & 3, or the Log. of 1,5	405465108108165
A + B + C + D + E + F + G A — B + C — D + E — F + G $= A + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \frac{2}{11} + \frac{2}{13}$		143841036225890 287682072451780

* * *

ARTICLE XX.

FOUR PAPERS.

By Mr. THOMAS KNIGHT, *Papcastle.*

No. I. Of the summation of the series represented by the expression

$$(s) \ldots\ldots \int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^{n})^{q}}.$$

It is well known that the most elegant method of summing complex series is by comparison with others of a simpler kind. Those made by the earlier analysts for this purpose were confined to * circular and hyperbolic series, or circular arcs and logarithms. This method of summation has of late been very much extended by Mr. *Spence*,† and a still larger class of simple series, than those he has so ingeniously treated of, will be necessary for our present purpose.

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* *Demoivre*, Miscell. Analyt. p. 111.

† Essay on Logarithmic Transcendents.

1. Let $\overset{r}{\underset{n,m}{\mathrm{F}}}(x)$ represent the series $\frac{x^m}{m^r} - \frac{x^{m+n}}{(m+n)^r} + \frac{x^{m+2n}}{(m+2n)^r} - \frac{x^{m+3n}}{(m+3n)^r} +$, then $\overset{r}{\underset{2,1}{\mathrm{F}}}(x)$ will denote the function $\overset{r}{\mathrm{C}}(x)$ of Mr. *Spence*, and $\overset{r}{\underset{1,1}{\mathrm{F}}}(x)$ will be his $\overset{r}{\mathrm{L}}(1+x)$. We will first shew how to find the sum of the series $\overset{r}{\underset{n,m}{\mathrm{F}}}(x)$ when x is positive and greater than unity:

$$\frac{1}{a^n+x^n} = \frac{1}{a^n}\left(1 - \frac{x^n}{a^n+x^n}\right) = \frac{1}{a^n}\left(1 - \frac{1}{a^n}\left(x^n - \frac{x^{2n}}{a^n+x^n}\right)\right) = \frac{1}{a^n}\left(1 - \frac{1}{a^n}\left(x^n - \frac{1}{a^n}\left(x^{2n} - \frac{x^{3n}}{a^n+x^n}\right)\right)\right) = \&\text{c.}$$

whence $\frac{1}{a^n+x^n} = \frac{1}{a^n} - \frac{x^n}{a^{2n}} + \frac{x^{2n}}{a^{3n}} - \frac{x^{3n}}{a^{4n}} +$, and by changing x into a, and the reverse,

$\frac{1}{a^n+x^n} = \frac{1}{x^n} - \frac{a^n}{x^{2n}} + \frac{a^{2n}}{x^{3n}} - \frac{a^{3n}}{x^{4n}} +$; whence, making $a = 1$, multiplying these expressions by $x^{m-1}dx$, and integrating, we have

$$\int \frac{x^{m-1}dx}{1+x^n} = \frac{x^m}{m} - \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} - \&\text{c.} = \overset{1}{\underset{n,m}{\mathrm{F}}}(x), \text{ or}$$

$\int \frac{x^{m-1}dx}{1+x^n} = -\frac{1}{(n-m)x^{n-m}} + \frac{1}{(2n-m)x^{2n-m}} - \frac{1}{(3n-m)x^{3n-m}} + \text{\&c.} = \overset{1}{\underset{n,m}{\varphi}}(x)$, if we denote by $\overset{r}{\underset{n,m}{\varphi}}(x)$ the function $\pm \left\{ \frac{1}{(n-m)^r x^{n-m}} - \frac{1}{(2n-m)^r x^{2n-m}} + \frac{1}{(3n-m)^r x^{3n-m}} - \right\}$, the upper sign having place when r is even. We have, evidently,

$$\int \frac{dx}{x} \times \overset{1}{\underset{n,m}{\text{F}}}(x) = \overset{2}{\underset{n,m}{\text{F}}}(x);\ \int \frac{dx}{x} \times \overset{2}{\underset{n,m}{\text{F}}}(x) = \overset{3}{\underset{n,m}{\text{F}}}(x);\ \ldots\ldots \int \frac{dx}{x} \times \overset{r}{\underset{n,m}{\text{F}}}(x) = \overset{r+1}{\underset{n,m}{\text{F}}}(x).$$

$$\int \frac{dx}{x} \times \overset{1}{\underset{n,m}{\varphi}}(x) = \overset{2}{\underset{n,m}{\varphi}}(x);\ \int \frac{dx}{x} \times \overset{2}{\underset{n,m}{\varphi}}(x) = \overset{3}{\underset{n,m}{\varphi}}(x);\ \ldots\ldots \int \frac{dx}{x} \times \overset{r}{\underset{n,m}{\varphi}}(x) = \overset{r+1}{\underset{n,m}{\varphi}}(x).$$

If we compare the two values of $\int \frac{x^{m-1}dx}{1+x^n}$, there arises

$$\overset{1}{\underset{n,m}{\text{F}}}(x) = \overset{1}{\underset{n,m}{\varphi}}(x) + \overset{\prime}{\text{C}}, \qquad \text{where} \qquad \overset{\prime}{\text{C}} = \overset{1}{\underset{n,m}{\text{F}}}(1) - \overset{1}{\underset{n,m}{\varphi}}(1),$$

$$\overset{2}{\underset{n,m}{\text{F}}}(x) = \overset{2}{\underset{n,m}{\varphi}}(x) + \overset{\prime}{\text{C}}\text{L}x + \overset{\prime\prime}{\text{C}}, \qquad \overset{\prime\prime}{\text{C}} = \overset{2}{\underset{n,m}{\text{F}}}(1) - \overset{2}{\underset{n,m}{\varphi}}(1),$$

$$\underset{n,m}{\overset{3}{\mathrm{F}}}(x) = \underset{n,m}{\overset{3}{\varphi}}(x) + \frac{\overset{\prime}{\mathrm{C}}}{2}(\mathrm{L}x)^2 + \overset{\prime\prime}{\mathrm{C}}\mathrm{L}x + \overset{\prime\prime\prime}{\mathrm{C}}, \qquad \overset{\prime\prime\prime}{\mathrm{C}} = \underset{n,m}{\overset{3}{\mathrm{F}}}(1) - \underset{n,m}{\overset{3}{\varphi}}(1),$$

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$$\underset{n,m}{\overset{r}{\mathrm{F}}}(x) = \underset{n,m}{\overset{r}{\varphi}}(x) + \frac{\overset{\prime}{\mathrm{C}}}{1.2.3\ldots(r-1)}(\mathrm{L}x)^{r-1} + \frac{\overset{\prime\prime}{\mathrm{C}}}{1.2.3\ldots(r-2)}(\mathrm{L}x)^{r-2} + \ldots + \overset{\prime\prime\ldots r}{\mathrm{C}}, \qquad \overset{\prime\prime\ldots r}{\mathrm{C}} = \underset{n,m}{\overset{r}{\mathrm{F}}}(1) - \underset{n,m}{\overset{r}{\varphi}}(1).$$

The valuation here given fails in the case of $\underset{n,an}{\overset{r}{\mathrm{F}}}(x)$, a being a whole positive number. To this case, however, a slight change would adapt it, if necessary; but this function is given by Mr. *Spence*'s Logarithmic Transcendents: for

$$\underset{n,an}{\overset{r}{\mathrm{F}}}(x) = \pm \left\{ \frac{1}{n^r} \times \underset{1,1}{\overset{r}{\mathrm{F}}}(x^n) - \frac{x^n}{n^r} + \frac{x^{2n}}{(2n)^r} - \ldots \pm \frac{x^{an-n}}{(an-n)^r} \right\}.$$

2. All that remains for us to do is to find finite expressions for $\overset{\prime}{\mathrm{C}}$, $\overset{\prime\prime}{\mathrm{C}}$, $\overset{\prime\prime\prime}{\mathrm{C}}$, &c. Now $\underset{n,m}{\overset{r}{\mathrm{F}}}(1) - \underset{n,m}{\overset{r}{\varphi}}(1) =$

$$\frac{1}{m^r} \pm \frac{1}{(2n-m)^r} + \frac{1}{(2n+m)^r} \pm \frac{1}{(4n-m)^r} + \frac{1}{(4n+m)^r} \pm$$

$$\mp \frac{1}{(n-m)^r} - \frac{1}{(n+m)^r} \mp \frac{1}{(3n-m)^r} - \frac{1}{(3n+m)^r} \mp$$

The upper signs having place when r is even. If, then, k denoting tangent $\frac{m\pi}{2n}$, we put

$$\overset{\prime}{P} = \frac{\pi}{2nk}, \qquad \overset{\prime}{Q} = \frac{k\pi}{2n},$$

$$\overset{\prime\prime}{P} = \frac{(2+2k^2)\pi^2}{2.4.n^2k^2} \qquad \overset{\prime\prime}{Q} = \frac{(2+2k^2)\pi^2}{2.4.n^2},$$

$$\overset{\prime\prime\prime}{P} = \frac{(6+6k^3)\pi^3}{2.4.6.n^3k^3} \qquad \overset{\prime\prime\prime}{Q} = \frac{(6k+6k^3)\pi^3}{2.4.6.n^3},$$

$$\overset{\prime\prime\prime\prime}{P} = \frac{(24+32k^2+8k^4)\pi^4}{2.4.6.8.n^4k^4} \qquad \overset{\prime\prime\prime\prime}{Q} = \frac{(8+32k^2+24k^4)\pi^4}{2.4.6.8.n^4},$$

&c. &c.

we shall have (Euleri. "Anal. infin." Tom. I. pp. 135, 136)

$$*\overset{\prime}{C} = \underset{n,m}{\overset{1}{F}}(1) - \underset{n,m}{\overset{1}{\varphi}}(1) = \overset{\prime}{P} + \overset{\prime}{Q}, \qquad \overset{\prime\prime}{C} = \underset{n,m}{\overset{2}{F}}(1) - \underset{n,m}{\overset{2}{\varphi}}(1) = \overset{\prime\prime}{P} - \overset{\prime\prime}{Q},$$

$$\overset{\prime\prime\prime}{C} = \underset{n,m}{\overset{3}{F}}(1) - \underset{n,m}{\overset{3}{\varphi}}(1) = \overset{\prime\prime\prime}{P} + \overset{\prime\prime\prime}{Q}, \qquad \overset{\prime\prime\prime\prime}{C} = \underset{n,m}{\overset{4}{F}}(1) - \underset{n,m}{\overset{4}{\varphi}}(1) = \overset{\prime\prime\prime\prime}{P} - \overset{\prime\prime\prime\prime}{Q},$$

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* We shall see hereafter that these constants may be found independently of circular functions, if that should be thought any advantage.

$$\overset{\prime\prime,\dots\overline{2r-1}}{C} = \underset{n,m}{\overset{2r}{F}}(1) - \underset{n,m}{\overset{2r-1}{\varphi}}(1) = \overset{\prime\prime,\dots\overline{2r-1}}{P} + \overset{\prime\prime,\dots\overline{2r-1}}{Q},$$

$$\overset{\prime\prime,\dots 2r}{C} = \underset{n,m}{\overset{2r}{F}}(1) - \underset{n,m}{\overset{2r}{\varphi}}(1) = \overset{\prime\prime,\dots 2r}{P} - \overset{\prime\prime,\dots 2r}{Q}.$$

If $k = \text{tangent}\ \frac{m\pi}{2n} = \pm 1$, $\overset{\prime\prime}{C}$, $\overset{\prime\prime\prime\prime}{C}, \dots\dots \overset{\prime\prime,\dots 2r}{C}$, vanish; as, for instance, in the case of $\underset{2m,m}{\overset{r}{F}}(x)$, which includes $\overset{r}{C}(x)$ of Mr. *Spence*. Let us now consider the summation of this class of series when x is equal (or nearly equal) to unity.

3. We easily find, by partial integrations, that

$$\int \frac{x^{r-1}\,dx}{(1+x^n)^s} = \frac{x^r}{r(1+x^n)^s} + \frac{snx^{r+n}}{r(r+n)(1+x^n)^{s+1}} + \frac{s(s+1)n^2x^{r+2n}}{r(r+n)(r+2n)(1+x^n)^{s+2}} + \dots\dots \dots\dots\dots\dots\dots (1).$$

Now, a being any whole positive number, we have, by division,

$$\int \frac{x^{m-1}\,dx}{1+x^n} = \frac{x^m}{m} - \frac{x^{n+m}}{n+m} + \frac{x^{2n+m}}{2n+m} - \dots\dots\dots \mp \frac{x^{(a-1)n+m}}{(a-1)n+m} \pm \int \frac{x^{an+m-1}\,dx}{1+x^n},$$ the upper signs having place when a is even. Make $r = an + m$, and, for brevity, put

$$\underset{n,m}{\overset{s}{f}}(x,a) = \frac{x^m}{m^s} - \frac{x^{n+m}}{(n+m)^s} + \frac{x^{2n+m}}{(2n+m)^s} - \dots\dots \mp \frac{x^{(a-1)n+m}}{((a-1)n+m)^s};$$ and we get the following equations

$\underset{n,m}{\overset{1}{F}}(x) = \underset{n,m}{\overset{1}{F}}(x,a) \pm \underset{n,r}{\overset{1}{F}}(x)$, $\underset{n,m}{\overset{2}{F}}(x) = \underset{n,m}{\overset{2}{F}}(x,a) \pm \underset{n,r}{\overset{2}{F}}(x)$, and generally $\underset{n,m}{\overset{s}{F}}(x) = \underset{n,m}{\overset{s}{F}}(x,a) \pm \underset{n,r}{\overset{s}{F}}(x)$. The only difficulty is in the valuation of the functions $\underset{n,r}{\overset{1}{F}}(x)$, $\underset{n,r}{\overset{2}{F}}(x)$, &c. We have, by equation (1),

$$\underset{n,r}{\overset{1}{F}}(x) = \frac{x^r}{r(1+x^n)} + \frac{nx^{r+n}}{r(r+n)(1+x^n)^2} + \frac{2n^2x^{r+2n}}{r(r+n)(r+2n)(1+x^n)^3} +,$$

and it will easily appear that, if $\underset{n,r}{\overset{s}{F}}(x)$ have the form

$$\overset{(s)}{\alpha} \times \frac{x^r}{r(1+x^n)} + \overset{(s)}{\beta} \times \frac{nx^{r+n}}{r(r+n)(1+x^n)^2} + \overset{(s)}{\gamma} \times \frac{2n^2x^{r+2n}}{r(r+n)(r+2n)(1+x^n)^3} +,$$

$\underset{n,r}{\overset{n+r}{F}}(x)$ will have exactly the same form, differing only in the coefficients $\overset{(s+1)}{\alpha}$, $\overset{(s+1)}{\beta}$, $\overset{(s+1)}{\gamma}$, &c., which are deduced from $\overset{(s)}{\alpha}$, $\overset{(s)}{\beta}$, $\overset{(s)}{\gamma}$, &c., by a simple and regular law; for if we multiply this value of $\underset{n,r}{\overset{s}{F}}(x)$ by $\frac{dx}{x}$, we find

$$\underset{n,r}{\overset{s+1}{F}}(x) = \frac{\overset{(s)}{\alpha}}{r}. \int \frac{x^{r-1}dx}{1+x^n} + \frac{\overset{(s)}{\beta}n}{r(r+n)}. \int \frac{x^{r+n-1}dx}{(1+x^n)^2} + \frac{\overset{(s)}{\gamma}2n^2}{r(r+n)(r+2n)}. \int \frac{x^{r+2n-1}dx}{(1+x^n)^3} + \frac{\overset{(s)}{\delta}2.3n^3}{r(r+n)(r+2n)(r+3n)}. \int \frac{x^{r+3n-1}dx}{(1+x^n)^4} + \&c.$$

and by substituting the values of the integrals, (given by equation (1)) which, for brevity, I omit, the form will be found the same as we assumed for $\overset{s}{\underset{n,r}{F}}(x)$, with the exception of the coefficients, which are formed from the former ones thus,

$\overset{(s+1)}{\alpha} = \frac{\overset{(s)}{\alpha}}{r}$, $\overset{(s+1)}{\beta} = \overset{(s+1)}{\alpha} + \frac{\overset{(s)}{\beta}}{r+n}$, $\overset{(s+1)}{\gamma} = \overset{(s+1)}{\beta} + \frac{\overset{(s)}{\gamma}}{r+2n}$, $\overset{(s+1)}{\delta} = \overset{(s+1)}{\gamma} + \frac{\overset{(s)}{\delta}}{r+3n}$, &c. We have seen that $\overset{(1)}{\alpha} = \overset{(1)}{\beta} = \overset{(1)}{\gamma} =$ &c. $= 1$, and from the equation $\overset{s}{\underset{n,m}{F}}(x) = \overset{s}{\underset{n,m}{F}}(x,a) \pm \overset{s}{\underset{n,r}{F}}(x)$, we derive, putting 1, 2, 3, &c. successively for s,

$$\frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - = \overset{1}{\underset{n,m}{F}}(1,a) \pm \left\{ \overset{(1)}{\alpha} \times \frac{1}{r.2} + \overset{(1)}{\beta} \times \frac{1.n}{r(r+n)2^2} + \overset{(1)}{\gamma} \times \frac{1.2.n^2}{r(r+n)(r+2n)2^3} + \right\},$$

$$\frac{1}{m^2} - \frac{1}{(m+n)^2} + \frac{1}{(m+2n)^2} - = \overset{2}{\underset{n,m}{F}}(1,a) \pm \left\{ \overset{(2)}{\alpha} \times \frac{1}{r.2} + \overset{(2)}{\beta} \times \frac{1.n}{r(r+n)2^2} + \overset{(2)}{\gamma} \times \frac{1.2.n^2}{r(r+n)(r+2n)2^3} + \right\},$$

$$\frac{1}{m^3} - \frac{1}{(m+n)^3} + \frac{1}{(m+2n)^3} - = \overset{3}{\underset{n,m}{F}}(1,a) \pm \left\{ \overset{(3)}{\alpha} \times \frac{1}{r.2} + \overset{(3)}{\beta} \times \frac{1.n}{r(r+n)2^2} + \overset{(3)}{\gamma} \times \frac{1.2.n^2}{r(r+n)(r+2n)2^3} + \right\},$$

&c. &c.

a may be taken greater or less according to the degree of convergency required. Nothing can exceed the simplicity of this method, and the simple series are found exactly in that order in which they enter into the complex series we are to treat of. The reader may make a trial of these formulas by finding Mr. *Spence*'s functions

$\overset{2}{L}(2)$, $\overset{3}{L}(2)$, $\overset{4}{L}(2)$, &c., and $\overset{1}{C}(1)$, $\overset{2}{C}(1)$, $\overset{3}{C}(1)$, &c. It is scarcely necessary to observe that, though we have, in the examples given above, made $x = 1$, the method is equally useful when x is *nearly* equal to unity.

We noticed, in Art. 2, that the constants $\overset{\prime}{c}$, $\overset{\prime\prime}{c}$, $\overset{\prime\prime\prime}{c}$, &c., might be found without the help of circular functions, for in the other series, which enters into those constants, viz.

$\frac{1}{(n-m)^r} - \frac{1}{(2n-m)^r} + \frac{1}{(3n-m)^r} - \frac{1}{(4n-m)^r} +$ some multiple of n will surpass m, suppose $3n$ to be this multiple,

and make $3n - m = \overset{\prime}{m}$; the series is transformed into $\frac{1}{(n-m)^r} - \frac{1}{(2n-m)^r} + \frac{1}{\overset{\prime}{m}{}^r} - \frac{1}{(\overset{\prime}{m}+n)^r} + \frac{1}{(\overset{\prime}{m}+2n)^r} -$ &c.

the latter part of which is similar to the series we have been treating of.

4. The successive summation of those series which come under the form*

$\frac{1}{q^t m^s} - \frac{1}{(q+n)^t (m+n)^s} + \frac{1}{(q+2n)^t (m+2n)^s} - \frac{1}{(q+3n)^t (m+3n)^s} +$ may be accomplished in a similar manner; for we have seen that

$$\frac{x^m}{m^s} - \frac{x^{m+n}}{(m+n)^s} + \frac{x^{m+2n}}{(m+2n)^s} - = \frac{x^m}{m^s} - \frac{x^{m+n}}{(m+n)^s} + \frac{x^{m+2n}}{(m+2n)^s} - \ldots\ldots \mp \frac{x^{m+(a-1)n}}{(m+(a-1)n)^s} \pm$$

$$\left\{ \overset{(s)}{\alpha} \times \frac{x^r}{r(1+x^n)} + \overset{(s)}{\beta} \times \frac{1.nx^{r+n}}{r(r+n)(1+x^n)^2} + \overset{(s)}{\gamma} \times \frac{1.2.n^2 x^{r+2n}}{r(r+n)(r+2n)(1+x^n)^3} + \right\}:$$

multiply this by $x^{p-1}dx$, and take the integral, putting $m + p = q$; then

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* See the note to No. 1.

$$\frac{x^q}{q.m^s} - \frac{x^{q+n}}{(q+n)(m+n)^s} + \frac{x^{q+2n}}{(q+2n)(m+2n)^s} - = \frac{x^q}{q.m^s} - \frac{x^{q+n}}{(q+n)(m+n)^s} + \ldots. \mp \frac{x^{q+(a-1)n}}{(q+(a-1)n)(m+(a-1)n)^s}$$

$$\pm \left\{ \frac{\overset{(s)}{\alpha}}{r} \times \int \frac{x^{r+p-1}dx}{(1+x^n)} + \frac{1.n\overset{(s)}{\beta}}{r(r+n)} \int \frac{x^{r+p+n-1}dx}{(1+x^n)^2} + \frac{1.2.n^2\overset{(s)}{\gamma}}{r(r+n)(r+2n)} \int \frac{x^{r+p+2n-1}dx}{(1+x^n)^3} + \right\}.$$

Make now $r + p = \acute{r}$, and expand these integrals by equa. (1).

Put $\frac{x^q}{q^t.m^s} - \frac{x^{q+n}}{(q+n)^t(m+n)^s} + \frac{x^{q+2n}}{(q+2n)^t(m+2n)^s} - = \overset{t,s}{\underset{n,(q,m)}{\mathrm{F}}}(x)$, and the finite expression

$$\frac{x^q}{q^s.m^s} - \frac{x^{q+n}}{(q+n)^t(m+n)^s} + \ldots. \mp \frac{x^{q+(a-1)n}}{(q+(a-1)n)^t(m+(a-1)n)^s} = \overset{t,s}{\underset{n,(q,m)}{\mathrm{F}}}(x,a), \text{ whence}$$

$\int \frac{dx}{x} \overset{t,s}{\underset{n,(q,m)}{\mathrm{F}}}(x) = \overset{t+1,s}{\underset{n,(q,m)}{\mathrm{F}}}(x)$, and $\int \frac{dx}{x} \overset{t,s}{\underset{n,(q,m)}{\mathrm{F}}}(x,a) = \overset{t+1,s}{\underset{n,(q,m)}{\mathrm{F}}}(x,a)$. By reasoning as we did in the former article, we easily see that, if

$\overset{(1)}{\mathrm{A}} = \frac{\overset{(s)}{\alpha}}{r}$, $\overset{(1)}{\mathrm{B}} = \overset{(1)}{\mathrm{A}} + \frac{\acute{r}\overset{(s)}{\beta}}{r(r+n)}$, $\overset{(1)}{\mathrm{C}} = \overset{(1)}{\mathrm{B}} + \frac{\acute{r}(\acute{r}+n)\overset{(s)}{\gamma}}{r(r+n)(r+2n)}$, &c. and afterwards

$$\overset{(t+1)}{\mathrm{A}} = \frac{\overset{(t)}{\mathrm{A}}}{\acute{r}}, \quad \overset{(t+1)}{\mathrm{B}} = \overset{(t+1)}{\mathrm{A}} + \frac{\overset{(t)}{\mathrm{B}}}{\acute{r}+n}, \quad \overset{(t+1)}{\mathrm{C}} = \overset{(t+1)}{\mathrm{B}} + \frac{\overset{(t)}{\mathrm{C}}}{\acute{r}+2n}, \quad \overset{(t+1)}{\mathrm{D}} = \overset{(t+1)}{\mathrm{C}} + \frac{\overset{(t)}{\mathrm{D}}}{\acute{r}+3n}, \text{ \&c.}$$

Then will

$$\frac{1}{q.m^{s}}-\frac{1}{(q+n)(m+n)^{s}}+\frac{1}{(q+2n)(m+2n)^{s}}-=\overset{1,s}{\underset{n,(q,m)}{F}}(1,a)\pm\left\{\overset{(1)}{A}\times\frac{1}{r'.2}+\overset{(1)}{B}\times\frac{1.n}{r'(r'+n)2^{2}}+\overset{(1)}{C}\times\frac{1.2.n^{2}}{r'(r'+n)(r'+2n)2^{3}}+\right\}$$

$$\frac{1}{q^{2}.m^{s}}+\frac{1}{(q+n)^{2}(m+n)^{s}}+\frac{1}{(q+2n)^{2}(m+2n)^{s}}-=\overset{2,s}{\underset{n,(q,m)}{F}}(1,a)\pm\left\{\overset{(2)}{A}\times\frac{1}{r'.2}+\overset{(2)}{B}\times\frac{1.n}{r'(r'+n)2^{2}}+\overset{(2)}{C}\times\frac{1.2.n^{2}}{r'(r'+n)(r'+2n)2^{3}}+\right\}$$

$$\frac{1}{q^{3}.m^{s}}-\frac{1}{(q+n)^{3}(m+n)^{s}}+\frac{1}{(q+2n)^{3}(m+2n)^{s}}-=\overset{3,s}{\underset{n,(q,m)}{F}}(1,a)\pm\left\{\overset{(3)}{A}\times\frac{1}{r'.2}+\overset{(3)}{B}\times\frac{1.n}{r'(r'+n)2^{2}}+\overset{(3)}{C}\times\frac{1.2.n^{2}}{r'(r'+n)(r'+2n)2^{3}}+\right\}$$

&c. &c.

If we put $q+v=w$, $r'+v=r''$, The successive summation of series of the form

$$\frac{1}{w^{e}.q^{t}.m^{s}}-\frac{1}{(w+n)^{e}(q+n)^{t}(m+n)^{s}}+\frac{1}{(w+2n)^{e}(q+2n)^{t}(m+2n)^{s}}-$$ &c. is evidently effected by formulas exactly similar to the preceding ones; the coefficients being formed in the same manner from those of the foregoing set.

5. The series we have hitherto treated of may be considered as simple ones: We shall now treat of the class of complex series of which the paper proposes to estimate the value. The case we are going to begin with will not completely solve the problem; but it is necessary for the demonstration of a property to be given hereafter.

LEMMA.

$$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+rn-1}dx}{(1+x^{n})^{r+1}}=\frac{m+(r-1)n}{r.n}\times(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+(r-1)n-1}dx}{(1+x^{n})^{r}}-$$

$\frac{1}{r.n}\times(s-1)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+(r-1)n-1}dx}{(1+x^{n})^{r}}$, where (s) and $(s-1)$ mark the number of the signs of integration. The demonstration of this, may be found in most books which treat on fluxions.

We get, by partial integration, $\int\frac{x^{m-1}dx}{1+x^{n}}=\frac{x^{m}}{m(1+x^{n})}+\frac{n}{m}\int\frac{x^{m+n-1}dx}{(1+x^{n})^{2}}$; whence, by transposition

$$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+n-1}dx}{(1+x^{n})^{2}}=\frac{m}{n}\times\overset{s}{\underset{n,m}{F}}(x)-\frac{1}{n}\times\overset{s-1}{\underset{n,m}{F}}(x).$$

Make, in the preceding lemma, $r=2$, 3, 4, &c. successively, and there will arise from its continual application, beginning with the last equation.

$$(s)\ldots\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+n-1}dx}{(1+x^{n})^{2}}=\frac{1}{n}\left\{m\overset{s}{\underset{n,m}{F}}(x)-\overset{s-1}{\underset{n,m}{F}}(x)\right\},$$

$$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+2n-1}dx}{(1+x^{n})^{3}}=\frac{1}{2.n^{2}}\left\{m(m+n)\overset{s}{\underset{n,m}{F}}(x)-(m+(m+n))\overset{s-1}{\underset{n,m}{F}}(x)+\overset{s-2}{\underset{n,m}{F}}(x)\right\},\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+3n-1}dx}{(1+x^{n})^{4}}=\frac{1}{2.3.n^{3}}\Big\{m(m+n)(m+2n)\overset{s}{\underset{n,m}{\mathrm{F}}}(x)-(m(m+n)+m(m+2n)+(m+n)(m+2n))\overset{s-1}{\underset{n,m}{\mathrm{F}}}(x)+(m+(m+n)+(m+2n))\overset{s-2}{\underset{n,m}{\mathrm{F}}}-\overset{s-3}{\underset{n,m}{\mathrm{F}}}(x)\Big\}.$$

&c. &c.

Here the law and its continuation are quite evident from the manner of formation by the Lemma.

The exponents above the F are not supposed to be confined to positive numbers, but must equally include negative ones, in order that the expressions may be sufficiently general; only we must observe that

$$\overset{0}{\underset{n,m}{\mathrm{F}}}(x)=\frac{x^{m}}{1+x^{n}},\quad\overset{-1}{\underset{n,m}{\mathrm{F}}}(x)=d.\frac{x^{m}}{1+x^{n}}\div\frac{dx}{x},\ \&c.$$

$$\text{Thus }\int\frac{dx}{x}\int\frac{x^{m+3n-1}dx}{(1+x^{n})^{4}}=\frac{1}{2.3.n^{3}}\Big\{m(m+n)(m+2n)\overset{2}{\underset{n,m}{\mathrm{F}}}(x)-(m(m+n)+m(m+2n)+(m+n)(m+2n))\overset{1}{\underset{n,m}{\mathrm{F}}}(x)$$

$$+(m+(m+n)+(m+2n))\overset{0}{\underset{n,m}{\mathrm{F}}}(x)-\overset{-1}{\underset{n,m}{\mathrm{F}}}(x)\Big\}=\frac{1}{2.3.n^{3}}\Big\{m(m+n)(m+2n)\overset{2}{\underset{n,m}{\mathrm{F}}}(x)-(m(m+n)+m(m+2n)+(m+n)(m+2n))\overset{1}{\underset{n,m}{\mathrm{F}}}(x)$$

$$+\big((m+n)+(m+2n)\big)\frac{x^{m}}{1+x^{n}}+\frac{nx^{m+n}}{(1+x^{n})^{2}}\Big\}.$$

Here follow a few simple examples of such series as require only the functions of Mr. *Spence.*

$\frac{1}{2^2} - \frac{2}{3^2} + \frac{3}{4^2} - \frac{4}{5^2} + = \overset{2}{\underset{1,1}{F}}(1) - \overset{1}{\underset{1,1}{F}}(1)$; $\frac{1}{3^2} - \frac{3}{4^2} + \frac{6}{5^2} - \frac{10}{6^2} + = \overset{2}{\underset{1,1}{F}}(1) - \frac{3}{2} \cdot \overset{1}{\underset{1,1}{F}}(1) + \frac{1}{4}$;

$\frac{1}{4^2} - \frac{4}{5^2} + \frac{10}{6^2} - \frac{20}{7^2} + = \overset{2}{\underset{1,1}{F}}(1) - \frac{11}{6} \cdot \overset{1}{\underset{1,1}{F}}(1) + \frac{11}{24}$; &c. all given by circular arcs and logarithms.

$\frac{1}{2^3} - \frac{2}{3^3} + \frac{3}{4^3} - \frac{4}{5^3} + = \overset{3}{\underset{1,1}{F}}(1) - \overset{2}{\underset{1,1}{F}}(1)$; $\frac{1}{3^3} - \frac{3}{4^3} + \frac{6}{5^3} - \frac{10}{6^3} + = \overset{3}{\underset{1,1}{F}}(1) - \frac{3}{2} \cdot \overset{2}{\underset{1,1}{F}}(1) + \frac{1}{2} \cdot \overset{1}{\underset{1,1}{F}}(1)$;

$\frac{1}{4^3} - \frac{4}{5^3} + \frac{10}{6^3} - \frac{20}{7^3} + = \overset{3}{\underset{1,1}{F}}(1) - \frac{11}{6} \cdot \overset{2}{\underset{1,1}{F}}(1) + \overset{1}{\underset{1,1}{F}}(1) - \frac{1}{12}$; &c. The following examples require the functions denoted by $\overset{r}{c}(1)$ in Mr. *Spence*'s work.

$\frac{1}{3^2} - \frac{2}{5^2} + \frac{3}{7^2} - \frac{4}{9^2} + = \frac{1}{2}\left\{\overset{2}{\underset{2,1}{F}}(1) - \overset{1}{\underset{2,1}{F}}(1)\right\}$; $\frac{1}{5^2} - \frac{3}{7^2} + \frac{6}{9^2} - \frac{10}{11^2} + = \frac{3}{8} \cdot \overset{2}{\underset{2,1}{F}}(1) - \frac{1}{2} \cdot \overset{1}{\underset{2,1}{F}}(1) + \frac{1}{16}$;

$\frac{1}{7^2} - \frac{4}{9^2} + \frac{10}{11^2} - \frac{20}{13^2} + = \frac{5}{16} \cdot \overset{2}{\underset{2,1}{F}}(1) - \frac{23}{48} \cdot \overset{1}{\underset{2,1}{F}}(1) + \frac{9}{16}$ &c.

$\frac{1}{3^3} - \frac{2}{5^3} + \frac{3}{7^3} - \frac{4}{9^3} + = \frac{1}{2}\left\{\overset{3}{\underset{2,1}{F}}(1) - \overset{2}{\underset{2,1}{F}}(1)\right\}$; $\frac{1}{5^3} - \frac{3}{7^3} + \frac{6}{9^3} - \frac{10}{11^3} + = \frac{3}{8} \cdot \overset{3}{\underset{2,1}{F}}(1) - \frac{1}{2} \overset{2}{\underset{2,1}{F}}(1) + \frac{1}{8} \overset{1}{\underset{2,1}{F}}(1)$;

$\frac{1}{7^3} - \frac{4}{9^3} + \frac{10}{11^3} - \frac{20}{13^3} + = \frac{5}{16} \cdot \overset{3}{\underset{2,1}{F}}(1) - \frac{23}{48} \cdot \overset{2}{\underset{2,1}{F}}(1) + \frac{9}{48} \cdot \overset{1}{\underset{2,1}{F}}(1) - \frac{1}{96}$; &c.

6. We denoted $(t)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int x^{p-1}\,dx \times \underset{n,m}{\overset{s}{\mathrm{F}}}(x)$ by $\underset{n,(q,m)}{\overset{t,s}{\mathrm{F}}}(x)$, where $q = m + p$.

Equations (2) may therefore be extended by the substitution of the latter function, we have successively

$$(t)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int x^{p-1}dx \times (s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+n-1}dx}{(1+x^n)^2} = \frac{1}{n}\left\{m\underset{n,(q,m)}{\overset{t,s}{\mathrm{F}}}(x) - \underset{n,(q,m)}{\overset{t,s-1}{\mathrm{F}}}(x)\right\} \ldots\ldots(3)$$

$$(t)\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int x^{p-1}dx \times (s)\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m+2n-1}dx}{(1+x^n)^2} = \frac{1}{2.n^2}\left\{m(m+n)\underset{n,(q,m)}{\overset{t,s}{\mathrm{F}}}(x) - (m+(m+n))\underset{n,(q,m)}{\overset{t,s-1}{\mathrm{F}}}(x) + \underset{n,(q,m)}{\overset{t,s-2}{\mathrm{F}}}(x)\right\}$$

&c. &c.

The extension of these equations still further, is sufficiently easy if it should be required.

7. We now proceed to the *general* solution of the problem at first proposed: for which purpose, we have occasion for a second

LEMMA.

$$(s)\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^n)^{r+1}} = \frac{1}{rn}\times(s-1)\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^n)^r} - \frac{m-rn}{rn}\times(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^n)^r}.$$

We easily find, by partial integration, that

$\int\frac{x^{m-1}dx}{(1+x^n)^2} = \frac{x^m}{n(1+x^n)} - \frac{m-n}{n}\int\frac{x^{m-1}dx}{1+x^n}$; indeed it results from the preceding equation. Hence

$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^n)^2}=-\frac{m-n}{n}\cdot\overset{s}{\underset{n,m}{F}}(x)+\frac{1}{n}\cdot\overset{s-1}{\underset{n,m}{F}}(x)$; and by a continual application of the Lemma, making, successively, $r=2, 3, 4,$ &c. we get the following set of equations

$$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^n)^2}=-\frac{m-n}{n}\cdot\overset{s}{\underset{n,m}{F}}(x)+\frac{1}{n}\cdot\overset{s-1}{\underset{n,m}{F}}(x),$$

$$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^n)^3}=\frac{(m-2n)(m-n)}{1.2.n^2}\cdot\overset{s}{\underset{n,m}{F}}(x)-\frac{(m-2n)+(m-n)}{1.2.n^2}\cdot\overset{s-1}{\underset{n,m}{F}}(x)+\frac{1}{1.2.n^2}\cdot\overset{s-2}{\underset{n,m}{F}}(x),$$

$$(s)\ldots\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1+x^n)^4}=-\frac{(m-3n)(m-2n)(m-n)}{1.2.3.n^3}\cdot\overset{s}{\underset{n,m}{F}}(x)+\frac{(m-3n)(m-2n)+(m-3n)(m-n)+(m-2n)(m-n)}{1.2.3.n^3}\cdot\overset{s-1}{\underset{n.m}{F}}(x)$$
$$-\frac{(m-3n)+(m-2n)+(m-n)}{1.2.3.n^2}\cdot\overset{s-2}{\underset{n,m}{F}}(x)+\frac{1}{1.2.3.n^3}\cdot\overset{s-3}{\underset{n,m}{F}}(x),\ldots\ldots\ldots\ldots(4)$$

&c. &c.

The exponents over the F, as we observed with respect to equations (2), must not be confined to positive numbers, but we must take for $\overset{0}{\underset{n,m}{F}}(x)$, $\overset{-1}{\underset{n,m}{F}}(x)$, $\overset{-2}{\underset{n,m}{F}}(x)$, &c. the functions there directed.

In like manner let $\overset{r}{\underset{n,m}{B}}(x)$ represent $\frac{x^m}{m^r}+\frac{x^{m+n}}{(m+n)^r}+\frac{x^{m+2n}}{(m+2n)^r}+$ &c.,

We find the following similar set of equations.

$$(s)\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1-x^n)^2}=-\frac{m-n}{n}\cdot\overset{s}{\underset{n,m}{\mathrm{B}}}(x)+\frac{1}{n}\cdot\overset{s-1}{\underset{n,m}{\mathrm{B}}}(x),\ \ldots\ldots\ldots\ldots\ldots(5)$$

$$(s)\ldots\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{x^{m-1}dx}{(1-x^n)^3}=\frac{(m-2n)(m-n)}{1.2.n^2}\cdot\overset{s}{\underset{n,m}{\mathrm{B}}}(x)-\frac{(m-2n)+(m-n)}{1.2.n^2}\cdot\overset{s-1}{\underset{n,m}{\mathrm{B}}}(x)+\frac{1}{1.2.n^2}\cdot\overset{s-2}{\underset{n,m}{\mathrm{B}}}(x),$$

&c. &c.

8. A few simple examples of the equations marked (4) and (5) follow.

The B has place along with the upper signs, the F with the lower.

$m=1,\ n=1,\ x=1,$

$$\frac{1}{1^s}\pm\frac{2}{2^s}+\frac{3}{3^s}\pm\frac{4}{4^s}+=\overset{s-1}{\underset{1,1}{\mathrm{B}}}(1),\text{ or }\overset{s-1}{\underset{1,1}{\mathrm{F}}}(1);$$

$$\frac{1}{1^s}\pm\frac{3}{2^s}+\frac{6}{3^s}\pm\frac{10}{4^s}+=\frac{1}{2}\left\{\overset{s-1}{\underset{1,1}{\mathrm{B}}}(1)+\overset{s-2}{\underset{1,1}{\mathrm{B}}}(1)\right\},\text{ or }\frac{1}{2}\left\{\overset{s-1}{\underset{1,1}{\mathrm{F}}}(1)+\overset{s-2}{\underset{1,1}{\mathrm{F}}}(1)\right\};$$

$$\frac{1}{1^s}\pm\frac{4}{2^s}+\frac{10}{3^s}\pm\frac{20}{4^s}+=\frac{1}{3}\cdot\overset{s-1}{\underset{1,1}{\mathrm{B}}}(1)+\frac{1}{2}\cdot\overset{s-2}{\underset{1,1}{\mathrm{B}}}(1)+\frac{1}{6}\cdot\overset{s-3}{\underset{1,1}{\mathrm{B}}}(1),\text{ or }\frac{1}{3}\cdot\overset{s-1}{\underset{1,1}{\mathrm{F}}}(1)+\frac{1}{2}\cdot\overset{s-2}{\underset{1.1}{\mathrm{F}}}(1)+\frac{1}{6}\cdot\overset{s-3}{\underset{1,1}{\mathrm{F}}}(1);$$

&c. &c. &c.

$m = 2,\ n = 1,\ x = 1.$

$$\frac{1}{2^s} \pm \frac{2}{3^s} + \frac{3}{4^s} \pm \frac{4}{5^s} + = -\overset{s}{\underset{1,2}{B}}(1) + \overset{s-1}{\underset{1,2}{B}}(1), \text{ or } -\overset{s}{\underset{1,2}{F}}(1) + \overset{s-1}{\underset{1,2}{F}}(1);$$

$$\frac{1}{2^s} \pm \frac{3}{3^s} + \frac{6}{4^s} \pm \frac{10}{5^s} + = -\frac{1}{2}\cdot\overset{s-1}{\underset{1,2}{B}}(1) + \frac{1}{2}\cdot\overset{s-2}{\underset{1,2}{B}}(1), \text{ or } -\frac{1}{2}\cdot\overset{s-1}{\underset{1,2}{F}}(1) + \frac{1}{2}\cdot\overset{s-2}{\underset{1,2}{F}}(1);$$

$$\frac{1}{2^s} \pm \frac{4}{3^s} + \frac{10}{4^s} \pm \frac{20}{5^s} + = -\frac{1}{6}\cdot\overset{s-1}{\underset{1,2}{B}}(1) + \frac{1}{6}\cdot\overset{s-3}{\underset{1,2}{B}}(1), \text{ or } -\frac{1}{6}\cdot\overset{s-1}{\underset{1,2}{F}}(1) + \frac{1}{6}\cdot\overset{s-3}{\underset{1,2}{F}}(1);$$

&c. &c. &c.

$m = 3,\ n = 1,\ x = 1.$

$$\frac{1}{3^s} \pm \frac{2}{4^s} + \frac{3}{5^s} \pm \frac{4}{6^s} + = -2\cdot\overset{s}{\underset{1,3}{B}}(1) + \overset{s-1}{\underset{1,3}{B}}(1), \text{ or } -2\cdot\overset{s}{\underset{1,3}{F}}(1) + \overset{s-1}{\underset{1,3}{F}}(1);$$

$$\frac{1}{3^s} \pm \frac{3}{4^s} + \frac{6}{5^s} \pm \frac{10}{6^s} + = \overset{s}{\underset{1,3}{B}}(1) - \frac{3}{2}\cdot\overset{s-1}{\underset{1,3}{B}}(1) + \frac{1}{2}\cdot\overset{s-2}{\underset{1,3}{B}}(1), \text{ or } \overset{s}{\underset{1,3}{F}}(1) - \frac{3}{2}\cdot\overset{s-1}{\underset{1,3}{F}}(1) + \frac{1}{2}\cdot\overset{s-2}{\underset{1,3}{F}}(1);$$

$$\frac{1}{3^s} \pm \frac{4}{4^s} + \frac{10}{5^s} \pm \frac{20}{6^s} + = \frac{1}{3}\cdot\overset{s-1}{\underset{1,3}{B}}(1) - \frac{1}{2}\cdot\overset{s-2}{\underset{1,3}{B}}(1) + \frac{1}{6}\cdot\overset{s-3}{\underset{1,3}{B}}(1), \text{ or } \frac{1}{3}\cdot\overset{s-1}{\underset{1,3}{F}}(1) - \frac{1}{2}\cdot\overset{s-2}{\underset{1,3}{F}}(1) + \frac{1}{6}\cdot\overset{s-3}{\underset{1,3}{F}}(1);$$

&c. &c. &c.

9. The numerical values of $\overset{s}{\underset{1,m}{B}}(1)$, when s does not exceed 20, may be found in *Euler's* "Calc. Diff." p. 456.

Let us try a single example of actual summation; and let the series be

$$\frac{1}{3^5}+\frac{4}{4^5}+\frac{10}{5^5}+\frac{20}{6^5}+=\frac{1}{3}\cdot\overset{4}{\underset{1,3}{\mathrm{B}}}(1)-\frac{1}{2}\cdot\overset{3}{\underset{1,3}{\mathrm{B}}}(1)+\frac{1}{6}\cdot\overset{2}{\underset{1,3}{\mathrm{B}}}(1);$$ now (Euleri, loc. citat.)

$$\frac{1}{3}\cdot\overset{4}{\underset{1,3}{\mathrm{B}}}(1)=\frac{1}{3}\Big)\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+(=\qquad ,3607744112-\frac{17}{48}$$

$$\frac{1}{6}\cdot\overset{2}{\underset{1,3}{\mathrm{B}}}(1)=\frac{1}{6}\Big)\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+(=\qquad ,2741556778-\frac{10}{48}$$

$$,6349300890-\frac{27}{48}$$

$$\frac{1}{2}\cdot\overset{3}{\underset{1,3}{\mathrm{B}}}(1)=\frac{1}{2}\Big)\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+(=\qquad ,6010284515-\frac{27}{48}$$

$$\frac{1}{3^5}+\frac{4}{4^5}+\frac{10}{5^5}+\frac{20}{6^5}+\&c.=\qquad ,0339016375$$

The series $\frac{1}{2^{2n+1}}\pm\frac{4}{3^{2n+1}}+\frac{10}{4^{2n+1}}\pm\frac{20}{5^{2n+1}}+=\frac{1}{6}\left\{\overset{2n-2}{\underset{1,2}{\mathrm{B}}}(1)-\overset{2n}{\underset{1,2}{\mathrm{B}}}(1)\right\}$, or $\frac{1}{6}\left\{\overset{2n-2}{\underset{1,2}{\mathrm{F}}}(1)-\overset{2n}{\underset{1,2}{\mathrm{F}}}(1)\right\}$; is remarkable, as it admits of summation, by means of the circumference of a circle, and the numbers of *Bernoulli*, exactly in those cases in which the simple series $\frac{1}{2^r}\pm\frac{1}{3^r}+\frac{1}{4^r}\pm\frac{1}{5^r}+$ &c. does not do so.

Thus, using the notation of *Euler*, and making $n = 2, 3, 4$, &c. successively

$$\frac{1}{2^5} + \frac{4}{3^5} + \frac{10}{4^5} + \frac{20}{5^5} + = \frac{1}{6}\left\{\frac{2^1 . \mathfrak{A}}{1.2} . \pi^2 - \frac{2^3 . \mathfrak{B}}{1.2.3.4} . \pi^4\right\};$$

$$\frac{1}{2^7} + \frac{4}{3^7} + \frac{10}{4^7} + \frac{20}{5^7} + = \frac{1}{6}\left\{\frac{2^3 . \mathfrak{B}}{1.2.3.4} . \pi^4 - \frac{2^5 \mathfrak{C}}{1.2.3\ldots 6} . \pi^6\right\},$$

$$\frac{1}{2^9} + \frac{4}{3^9} + \frac{10}{4^9} + \frac{20}{5^9} + = \frac{1}{6}\left\{\frac{2^5 . \mathfrak{C}}{1.2.3\ldots 6} . \pi^6 - \frac{2^7 . \mathfrak{D}}{1.2.3\ldots 8} . \pi^8\right\},$$

&c. &c.

I ought to have observed that equations (4) are capable of the same extension, as we gave (in equations (3)) to equations (2).

10. Put, in the first of equations (4), $m + n$ for m; in the second, $m + 2n$ for m; in the third, $m + 3n$ for m; &c. There will result

$$(s) \ldots . \int \frac{dx}{x} \int \frac{dx}{x} \int \frac{x^{m+n-1} dx}{(1+x^n)^2} = -\frac{m}{n} . \underset{n,m+n}{\overset{s}{\mathrm{F}}}(x) + \frac{1}{n} . \underset{n,m+n}{\overset{s-1}{\mathrm{F}}}(x);$$

$$(s) \ldots . \int \frac{dx}{x} \int \frac{dx}{x} \int \frac{x^{m+2n-1} dx}{(1+x^n)^3} = \frac{m(m+n)}{1.2.n^2} . \underset{n,m+2n}{\overset{s}{\mathrm{F}}}(x) - \frac{m+(m+n)}{1.2.n^2} . \underset{n,m+2n}{\overset{s-1}{\mathrm{F}}}(x) + \frac{1}{1.2.n^2} . \underset{n,m+2n}{\overset{s-2}{\mathrm{F}}}(x);$$

$$(s) \ldots . \int \frac{dx}{x} \int \frac{dx}{x} \int \frac{x^{m+3n-1} dx}{(1+x^n)^4} = -\frac{m(m+n)(m+2n)}{1.2.3.n^3} . \underset{n,m+3n}{\overset{s}{\mathrm{F}}}(x) + \frac{m(m+n)+m(m+2n)+(m+n)(m+2n)}{1.2.3.n^3} . \underset{n,m+3n}{\overset{s-1}{\mathrm{F}}}(x)$$
$$- \frac{m+(m+n)+(m+2n)}{1.2.3.n^3} . \underset{n,m+3n}{\overset{s-2}{\mathrm{F}}}(x) + \frac{1}{1.2.3.n^3} . \underset{n,m+3n}{\overset{s-3}{\mathrm{F}}}(x);$$

&c. &c.

If we compare these with equations (2), we shall get the following properties, characterizing the function $\overset{r}{\underset{n,m}{\mathrm{F}}}(x)$:

$$m\left\{\overset{s}{\underset{n,m}{\mathrm{F}}}(x)+\overset{s}{\underset{n,m+n}{\mathrm{F}}}(x)\right\}-\left\{\overset{s-1}{\underset{n,m}{\mathrm{F}}}(x)+\overset{s-1}{\underset{n,m+n}{\mathrm{F}}}(x)\right\}=0;$$

$$m(m+n)\left\{\overset{s}{\underset{n,m}{\mathrm{F}}}(x)-\overset{s}{\underset{n,m+2n}{\mathrm{F}}}(x)\right\}-(m+(m+n))\left\{\overset{s-1}{\underset{n,m}{\mathrm{F}}}(x)-\overset{s-1}{\underset{n,m+2n}{\mathrm{F}}}(x)\right\}+\overset{s-2}{\underset{n,m}{\mathrm{F}}}(x)-\overset{s-2}{\underset{n,m+2n}{\mathrm{F}}}(x)=0;$$

$$m(m+n)(m+2n)\left\{\overset{s}{\underset{n,m}{\mathrm{F}}}(x)+\overset{s}{\underset{n,m+3n}{\mathrm{F}}}(x)\right\}-(m(m+n)+m(m+2n)+(m+n)(m+2n))\left\{\overset{s-1}{\underset{n,m}{\mathrm{F}}}(x)+\overset{s-1}{\underset{n,m+3n}{\mathrm{F}}}(x)\right\}+$$

$$(m+(m+n)+(m+2n))\left\{\overset{s-2}{\underset{n,m}{\mathrm{F}}}(x)+\overset{s-2}{\underset{n,m+3n}{\mathrm{F}}}(x)\right\}-\left\{\overset{s-3}{\underset{n,m}{\mathrm{F}}}(x)+\overset{s-3}{\underset{n,m+3n}{\mathrm{F}}}(x)\right\}=0.$$

&c. &c.

Analogous properties, with respect to the function $\overset{r}{\underset{n,m}{\mathrm{B}}}(x)$, are

$$m\left\{\overset{s}{\underset{n,m}{\mathrm{B}}}(x)-\overset{s}{\underset{n,m+n}{\mathrm{B}}}(x)\right\}-\left\{\overset{s-1}{\underset{n,m}{\mathrm{B}}}(x)-\overset{s-1}{\underset{n,m+n}{\mathrm{B}}}(x)\right\}=0;$$

$$m(m+n)\left\{\overset{s}{\underset{n,m}{\mathrm{B}}}(x)-\overset{s}{\underset{n,m+2n}{\mathrm{B}}}(x)\right\}-(m+(m+n))\left\{\overset{s-1}{\underset{n,m}{\mathrm{B}}}(x)-\overset{s-1}{\underset{n,m+2n}{\mathrm{B}}}(x)\right\}+\overset{s-2}{\underset{n,m}{\mathrm{B}}}(x)-\overset{s-2}{\underset{n,m+2n}{\mathrm{B}}}(x)=0,^{*}$$

&c. &c.

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[*] These properties are, however, a simple Corollary to the Theory of Equations, and only remarkable here from the manner in which they have been found.

Note. To complete the theory of the function $\overset{s}{\underset{n,m}{F}}(x)$, it would be necessary to give formulas, by means of which we might ascend, from it, to $\overset{t,s}{\underset{n,(q,m)}{F}}(x)$, &c.; in imitation of *Mr. Spence*, p. 113, & seq. Nothing can be easier; Putting $m + p = q$, we have

$\overset{1,s}{\underset{n,(q,m)}{F}}(x) = \int \overset{s}{\underset{n,m}{F}}(x)\, x^{p-1}dx$, $\overset{2,s}{\underset{n,(q,m)}{F}}(x) = \int \frac{dx}{x} \overset{1,s}{\underset{n,(q,m)}{F}}(x)$, $\overset{3\,s}{\underset{n,(q,m)}{F}}(x) = \int \frac{dx}{x} \overset{2,s}{\underset{n,(q,m)}{F}}(x)$; Now, integrating by parts, we find

$$\left.\begin{aligned}
\int \overset{s}{\underset{n,m}{F}}(x)\, x^{p-1}\,dx = \overset{1,s}{\underset{n,(q,m)}{F}}(x) &= \overset{s}{\underset{n,m}{F}}(x)\frac{x^p}{p} - \overset{s-1}{\underset{n,m}{F}}(x)\frac{x^p}{p^2} + \overset{s-2}{\underset{n,m}{F}}(x)\frac{x^p}{p^3} - \ldots\ldots \pm \overset{1}{\underset{n,m}{F}}(x)\frac{x^p}{p^s} \mp \frac{1}{p^s}\cdot\overset{1}{\underset{n,q}{F}}(x),\\
-\int \overset{s-1}{\underset{n,m}{F}}(x)\, x^{p-1}dx &= -\overset{s-1}{\underset{n,m}{F}}(x)\frac{x^p}{p} + \overset{s-2}{\underset{n,m}{F}}(x)\frac{x^p}{p^2} - \ldots\ldots \pm \overset{1}{\underset{n,m}{F}}(x)\frac{x^p}{p^{s-1}} \mp \frac{1}{p^{s-1}}\cdot\overset{1}{\underset{n,q}{F}}(x)\\
\int \overset{s-2}{\underset{n,m}{F}}(x)\, x^{p-1}dx &= \overset{s-2}{\underset{n,m}{F}}(x)\frac{x^p}{p} - \ldots\ldots \pm \overset{1}{\underset{n,m}{F}}(x)\frac{x^p}{p^{s-2}} \mp \frac{1}{p^{s-2}}\cdot\overset{1}{\underset{n,q}{F}}(x)
\end{aligned}\right\}\begin{array}{l}\text{The upper}\\ \text{sign being}\\ \text{taken when}\\ s\text{ is odd.}\end{array}$$

&c. &c.

$$\int \frac{dx}{x}\overset{1,s}{\underset{n,(q,m)}{F}}(x) = \overset{2,s}{\underset{n,(q,m)}{F}}(x) = \int \overset{s}{\underset{n,m}{F}}(x)\frac{x^{p-1}dx}{p} - \int \overset{s-1}{\underset{n,m}{F}}(x)\frac{x^{p-1}dx}{p^2} + \int \overset{s-2}{\underset{n,m}{F}}(x)\frac{x^{p-1}dx}{p^3} - \ldots\ldots \pm \int \overset{1}{\underset{n,m}{F}}(x)\frac{x^{p-1}dx}{p^s} \mp \frac{1}{p^s}\overset{2}{\underset{n,q}{F}}(x);$$

or, by substituting the values found above,

$$\underset{n,(q,m)}{\overset{2,s}{\mathrm{F}}}(x) = \underset{n,m}{\overset{s}{\mathrm{F}}}(x)\frac{x^p}{p^2} - 2\underset{n,m}{\overset{s-1}{\mathrm{F}}}(x)\frac{x^p}{p^3} + 3\underset{n,m}{\overset{s-2}{\mathrm{F}}}(x)\frac{x^p}{p^4} - \dots\dots\dots \pm s\underset{n,m}{\overset{1}{\mathrm{F}}}(x)\frac{x^p}{p^{s+1}} \mp \frac{s}{p^{s+1}}\cdot\underset{n,q}{\overset{1}{\mathrm{F}}}(x) \mp \frac{1}{p^s}\cdot\underset{n,q}{\overset{2}{\mathrm{F}}}(x).$$

In like manner, $\int\frac{dx}{x}\underset{n,(q,m)}{\overset{2,s}{\mathrm{F}}}(x) =$

$$\underset{n,(q,m)}{\overset{3,s}{\mathrm{F}}}(x) = \underset{n,m}{\overset{s}{\mathrm{F}}}(x)\frac{x^p}{p^3} - 3\underset{n,m}{\overset{s-1}{\mathrm{F}}}(x)\frac{x^p}{p^4} + 6\underset{n,m}{\overset{s-2}{\mathrm{F}}}(x)\frac{x^p}{p^5} - \dots \pm \frac{s(s+1)}{1\cdot 2}\underset{n,m}{\overset{1}{\mathrm{F}}}(x)\frac{x^p}{p^{s+2}} \mp \frac{s(s+1)}{1.2.p^{s+2}}\cdot\underset{n,q}{\overset{1}{\mathrm{F}}}(x) \mp \frac{s}{p^{s+1}}\cdot\underset{n,q}{\overset{2}{\mathrm{F}}}(x) \mp \frac{1}{p^s}\cdot\underset{n,q}{\overset{3}{\mathrm{F}}}(x),$$

and, in general, as is plain from the manner of continuation, and the nature of the figurate numbers,

$$\underset{n,(q,m)}{\overset{t,s}{\mathrm{F}}}(x) = \underset{n,m}{\overset{s}{\mathrm{F}}}(x)\frac{x^p}{p^t} - t\underset{n,m}{\overset{s-1}{\mathrm{F}}}(x)\frac{x^p}{p^{t+1}} + \frac{t(t+1)}{1\cdot 2}\underset{n,m}{\overset{s-2}{\mathrm{F}}}(x)\frac{x^p}{p^{t+2}} - \dots\dots \pm \frac{t(t+1)\dots(t+s-2)}{1\cdot 2\cdot 3\dots(s-1)}\underset{n,m}{\overset{1}{\mathrm{F}}}(x)\frac{x^p}{p^{t+s-1}}$$

$$\mp\left\{\frac{1}{p^s}\cdot\underset{n,q}{\overset{t}{\mathrm{F}}}(x) + \frac{s}{p^{s+1}}\underset{n,q}{\overset{t-1}{\mathrm{F}}}(x) + \frac{s(s+1)}{1.2.p^{s+2}}\underset{n,q}{\overset{t-2}{\mathrm{F}}}(x) + \dots\dots + \frac{s(s+1)\dots(s+t-2)}{1\cdot 2\dots(t-1)p^{s+t-1}}\underset{n,q}{\overset{1}{\mathrm{F}}}(x)\right\} \dots\dots\dots\dots\dots(\alpha)$$

If, next, putting $q + v = w$, we put $\underset{n,(w,q,m)}{\overset{e,t,s}{\mathrm{F}}}(x) = \frac{x^w}{w^e\cdot q^t\cdot m^s} - \frac{x^{w+n}}{(w+n)^e\cdot(q+n)^t\cdot(m+n)^s} + \frac{x^{w+2n}}{(w+2n)^e\cdot(q+2n)^t\cdot(m+2n)^s} -,$

we have, in like manner,

$$\underset{n,(w,q,m)}{\overset{e,t,s}{F}}(x) = \underset{n,(q,m)}{\overset{t,s}{F}}(x)\frac{x^{v}}{v^{e}} - e\,\underset{n,(q,m)}{\overset{t-1,s}{F}}(x)\cdot\frac{x^{v}}{v^{e+1}} + \frac{e(e+1)}{1.2}\underset{n,(q,m)}{\overset{t-2,s}{F}}(x)\cdot\frac{x^{v}}{v^{e+2}} - \ldots\ldots \pm \frac{e(e+1)\ldots.(e+t-2)}{1.2.3\ldots.(t-1)}\underset{n,(q,m)}{\overset{1,s}{F}}(x)\frac{x^{v}}{p^{e+t-1}}$$

$$\mp\left\{\frac{1}{v^{t}}\underset{n,(w,m)}{\overset{e,s}{F}}(x) + \frac{t}{v^{t+1}}\underset{n,(w,m)}{\overset{e-1,s}{F}}(x) + \ldots\ldots\ldots + \frac{t(t+1)\ldots.(t+e-2)}{1.2\ldots.(e-1)v^{t+e-1}}\underset{n,(w,m)}{\overset{1,s}{F}}(x)\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta).$$

It is unnecessary to say that Mr. *Spence*'s forms, p. 114, & seq. are easily derived from (α) and (β). These expressions may; it is evident, be further extended under an exactly similar form.

No. II. A General Expression for $\int\varphi(x)d.\psi(x)$.

Let $\underset{1}{\chi}$, $\underset{2}{\chi}$, $\underset{3}{\chi}$, &c. be any functions of x, taken *ad libitum*; it is required to find another series of functions $\underset{1}{X}$, $\underset{2}{X}$, $\underset{3}{X}$, &c., such that $\int\varphi(x)\,d.\psi(x) = \varphi(x)\psi(x) + \underset{1}{X}.\underset{1}{\chi} + \underset{2}{X}.\underset{2}{\chi} + \underset{3}{X}.\underset{3}{\chi} + .$ If we take the differential of this equation, there arises

$$\varphi(x)\,d.\psi(.x) = \varphi(x)\,d.\psi(x) + d.\varphi(x).\psi(x) + d.\underset{1}{X}.\underset{1}{\chi} + d.\underset{2}{X}.\underset{2}{\chi} + \&c.$$
$$+ \underset{1}{X}.d.\underset{1}{\chi} + \underset{2}{X}.d.\underset{2}{\chi} + \underset{3}{X}.d.\underset{3}{\chi} + \&c.$$

Now we may satisfy this equation in any way we think proper, provided that we make the whole of the second member, with the exception of its first term, vanish. If then, we equal each perpendicular column, separately, to nothing, (denoting the differential coefficients, with respect to x, by strokes over the quantities) we find

$\varphi'(x)\psi(x) + X_1 . \chi'_1 = 0$, $X'_1 . \chi_1 + X_2 . \chi'_2 = 0$, ... $X'_n . \chi_n + X_{n+1} . \chi'_{n+1} = 0$; &c. or

(1).... $X_1 = -\frac{\varphi'(x)\psi(x)}{\chi'_1}$, $X_2 = -\frac{X'_1 . \chi_1}{\chi'_2}$, $X_3 = -\frac{X'_2 . \chi_2}{\chi'_3}$, ... $X_{n+1} = -\frac{X'_n . \chi_n}{\chi'_{n+1}}$, &c.

We may make the series terminate whenever we think proper; indeed this will *always* require to be done at the point where the terms would otherwise become infinite.

Suppose, for instance, we wish it to extend no further than to the term $X_{n+1} . \chi_{n+1}$; we have only to make $\chi'_{n+1} = X'_n . \chi_n$, which gives $X_{n+1} = -1$, $X_{n+2} = 0$, $X_{n+3} = 0$, &c.; and the last term of the series is $X_{n+1} . \chi_{n+1} = -\int X'_n . \chi_n \, dx$.

By this means, $\int \varphi(x)\, d\psi(x)$ is made to depend on a new integral, and the reverse; whence arise innumerable theorems for the comparison of integrals. The equations marked (1) give, successively,

$X_1 = -\frac{\varphi'(x)\psi(x)}{\chi'_1}$; $X_2 = \left(\frac{\varphi'(x)\psi(x)}{\chi'_1}\right)' \times \frac{\chi_1}{\chi'_2}$; $X_3 = -\left(\left(\frac{\varphi'(x)\psi(x)}{\chi'_1}\right)' \times \frac{\chi_1}{\chi'_2}\right)' \times \frac{\chi_2}{\chi'_3}$; &c.; whence

$$\int\varphi(x)\,d\,.\,\psi(x)=\varphi(x)\,\psi(x)-\frac{\varphi'(x)\,\psi(x)}{\chi'_1}\times\chi_1+\left(\frac{\varphi'(x)\,\psi(x)}{\chi'_1}\right)'\times\frac{\chi_1}{\chi'_2}\times\chi_2-$$

$$\left(\left(\frac{\varphi'(x)\,\psi(x)}{\chi'_1}\right)'\times\frac{\chi_1}{\chi'_2}\right)'\times\frac{\chi_2}{\chi'_3}\times\chi_3+\left(\left(\left(\frac{\varphi'(x)\,\psi(x)}{\chi'_1}\right)'\times\frac{\chi_1}{\chi'_2}\right)'\times\frac{\chi_2}{\chi'_3}\right)'\times\frac{\chi_3}{\chi'_4}\times\chi_4-\&c.\ldots.(2).$$

This series is the simplest that can be *generally* useful for the calculation of integrals. It may be proper, however, to shew how other series are derived from it. Put $\chi_2=\int\chi_1\,dx$, $\chi_3=\int dx\int\chi_1\,dx$, $\chi_4=\int dx\int dx\int\chi_1\,dx$, &c. and the above form becomes

$$\int\varphi(x)\,d\,.\,\psi(x)=\varphi(x)\,\psi(x)-\frac{\varphi'(x)\,\psi(x)}{\chi'_1}\times\chi_1+\left(\frac{\varphi'(x)\,\psi(x)}{\chi'_1}\right)'\int\chi_1\,dx-\left(\frac{\varphi'(x)\,\psi(x)}{\chi'_1}\right)''\int dx\int\chi_1\,dx+$$

which admits of further simplification in two ways, by making $\chi'_1=\varphi'(x)$, or $\chi'_1=\psi(x)$; we get thus

$$\left.\begin{array}{l}\int\varphi(x)\,d\,.\,\psi(x)=\varphi(x)\,\psi(x)-\varphi'(x)\int\psi(x)\,dx+\varphi''(x)\int dx\int\psi(x)\,dx-\&c.\\ \int\varphi(x)\,d\,.\,\psi(x)=\psi'(x)\int\varphi(x)\,dx-\psi''(x)\int dx\int\varphi(x)\,dx+\psi'''(x)\int dx\int dx\int\varphi(x)\,dx-\&c.\end{array}\right\}\ldots\ldots\ldots.(3).$$

These two series, however, have so little generality, that they will not give any *regular* form, unless the quantity $\varphi(x)\, d\psi(x)$ can be so divided that the successive differential coefficients, and integrals, in the two series, may each consist of only a single term. We may still further simplify equations (3) till they are, in fact, nearly useless. By making $\psi(x) = x$, in the first; and $\varphi(x) = x$ in the second, we get

$$\int \varphi(x)\, dx = \varphi(x) x - \frac{\overset{\prime}{\varphi}(x)}{2} x^2 + \frac{\overset{\prime\prime}{\varphi}(x)}{2 \cdot 3} x^3 - \frac{\overset{\prime\prime\prime}{\varphi}(x)}{2 \cdot 3 \cdot 4} x^4 + \text{\&c., and}$$

$\int x\, d\psi(x) = \overset{\prime}{\psi}(x)\, \frac{x^2}{1 \cdot 2} - \overset{\prime\prime}{\psi}(x)\, \frac{x^3}{1 \cdot 2 \cdot 3} + \overset{\prime\prime\prime}{\psi}(x)\, \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{\&c.}$ These are the two series given by *John Bernoulli*; (vide "Oper." ejus tom 2^dum^ p. 488, 489. *Bossut*, "Trait. de Calc. Different. &c." p. 444. *Arbogast*, "Du Calc. des Derivations," p. 334. The first only of these series is given by most authors; and it is of so little use, that even $\int (a + bx^n)^{\frac{p}{q}} dx$ cannot be found by it, if n be greater than unity. With respect to the use of equation (2). We may, in all cases, very easily see what functions must be taken for $\underset{1}{x}, \underset{2}{x}, \underset{3}{x}$ &c. in order to get a regular form.

It is required to find a series for $\int (\alpha + \beta x^r)^p (a + bx^r)^{n-1} x^{mr-1} dx$, without expanding either of the binomials. This is proposed merely as an example of the method, such series being by no means necessary. We shall begin with

$$\int (\alpha + \beta x^r)^p (a + bx^r)^{n-1} nbrx^{r-1} dx. \text{ Make } \varphi(x) = (\alpha + \beta x^r)^p,$$

$$\psi(x) = (a + bx^r)^n. \text{ Put } \underset{1}{x} = (a + bx^r)^{n+1},\ \underset{2}{x} = (a + bx^r)^{n+2}, \text{ \&c. then}$$

$$\frac{\psi(x)}{\overset{\prime}{x}_1} = \frac{1}{(n+1)brx^{r-1}}, \quad \frac{x_1}{\overset{\prime}{x}_2} = \frac{1}{(n+2)brx^{r-1}}, \quad \frac{x_2}{\overset{\prime}{x}_3} = \frac{1}{(n+3)brx^{r-1}}, \text{ \&c.}$$

$$\overset{\prime}{\varphi}(x) = p\beta r(\alpha+\beta x'^{r})^{p-1}x^{r-1}, \text{ whence } X_1 = -\frac{p\beta}{(n+1)b}(\alpha+\beta x'^{r})^{p-1},$$

$$X_2 = \frac{p(p-1)\beta^2}{(n+1)(n+2)b^2}(\alpha+\beta x'^{r})^{p-2}, \quad X_3 = -\frac{p(p-1)(p-2)\beta^3}{(n+1)(n+2)(n+3)b^3}(\alpha+\beta x'^{r})^{p-3}, \text{ \&c. and}$$

$$\int(\alpha+\beta x'^{r})^{p}(a+bx'^{r})^{n-1}x^{r-1}dx = \frac{1}{nbr}\Big\{(\alpha+\beta x'^{r})^{p}(a+bx'^{r})^{n} - \frac{p\beta}{(n+1)b}(\alpha+\beta x'^{r})^{p-1}(a+bx'^{r})^{n+1}$$

$$+\frac{p(p-1)\beta^2}{(n+1)(n+2)b^2}(\alpha+\beta x'^{r})^{p-2}(a+bx'^{r})^{n+2} - \text{\&c.}\Big\}.$$ Put for brevity

$(\alpha+\beta x'^{r})^{p} = (\;)^{p}$, $(a+bx'^{r})^{n-1} = \{\;\}^{n-1}$; then by means of the expression just now found, we get, successively,

$$\int(\;)^{p}\{\;\}^{n-1}x^{r-1}dx = \frac{1}{n.r.b}\times(\;)^{p}\{\;\}^{n} - \frac{p\beta}{n(n+1)rb^2}(\;)^{p-1}\{\;\}^{n+1}$$

$$+\frac{p(p-1)\beta^2}{n(n+1)(n+2)rb^3}(\;)^{p-2}\{\;\}^{n+2} -, \ldots\ldots\ldots\ldots(3)$$

$$\int x^{r-1}dx\int(\,)^{p}\{\,\}^{n-1}x^{r-1}\,dx \quad = \frac{\}\,\{}{n(n+1)r^{2}b^{2}}(\,)^{p}\{\,\}^{n} - \frac{2p\beta\,\}\,\{}{n(n+1)(n+2)r^{2}b^{3}}(\,)^{p-1}\{\,\}^{n+1}$$

$$+\frac{3p(p-1)\beta^{2}\,\}\,\{}{n(n+1)(n+2)(n+3)r^{2}b^{4}}(\,)^{p-2}\{\,\}^{n+2} \quad -,\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\int x^{r-1}dx\int x^{r-1}dx\int(\,)^{p}\{\,\}^{n-1}x^{r-1}dx = \frac{\}\,\{^{2}}{n(n+1)(n+2)r^{2}b^{3}}(\,)^{p}\{\,\}^{n} - \frac{3p\beta\,\}\,\{^{2}}{n(n+1)\ldots(n+3)r^{3}b^{4}}(\,)^{p-1}\{\,\}^{n+1}$$

$$+\frac{6p(p-1)\beta^{2}\}\,\{^{2}}{n(n+1)\ldots.(n+4)r^{3}b^{5}}(\,)^{p-2}\{\,\}^{n+2} -,\ldots.\ \ldots\ldots\ldots\ldots\ldots(5)$$

&c. &c.

Now $\int(\alpha+\beta x^{r})^{p}(a+bx^{r})^{n-1}x^{mr-1}dx = \int(\beta+\alpha x^{-r})^{p}(b+ax^{-r})^{n-1}x^{-r-1}dx \times x^{(m+p+n)r}$,

put $\mu = m+p+n$, $(\beta+\alpha x^{-r})^{p} = \,)\,(^{p}$, $(b+ax^{-r})^{n-1} = \}\{^{n-1}$

Make $\phi(x) = x^{\mu r}$, $\psi(x) = \int\,)\,(^{p}\,\}\{^{n-1}\,x^{-r-1}dx$,

$\varkappa_{1} = \int x^{-r-1}dx\int\,)\,(^{p}\,\}\{^{n-1}\,x^{-r-1}dx$, $\varkappa_{2} = \int x^{-r-1}dx\int x^{-r-1}dx\int\,)\,(^{p}\,\}\{^{n-1}\,x^{-r-1}dx$, &c.

then $\dfrac{\psi(x)}{\varkappa'_{1}} = x^{r+1}$, $\dfrac{\varkappa_{1}}{\varkappa'_{1}} = x^{r+1}$, $\dfrac{\varkappa_{2}}{\varkappa'_{3}} = x^{r+1}$, &c.; whence

$\underset{1}{X} = -\mu r x^{\mu r + r}$, $\underset{2}{X} = \mu r(\mu r + r)x^{\mu r + 2r}$, $\underset{3}{X} = -\mu r(\mu r + r)(\mu r + 2r)x^{\mu r + 3r}$, &c. and

$$\int(\alpha + \beta x)^p (a + bx^r)^{n-1} x^{mr-1} dx = x^{\mu r}\int)(^p \}\{^{n-1} x^{-r-1} dx - \mu r x^{\mu r + r}\int x^{-r-1} dx \int)(^p \}\{^{n-1} x^{-r-1} dx$$

$$+ \mu r(\mu r + r)x^{\mu r + 2r}\int x^{-r-1} dx \int x^{-r-1} dx \int)(^p \}\{^{n-1} x^{-r-1} dx - \&c.$$

But these integrals are got from equations (3), (4), (5), &c. by putting $-r$ for r, a for b, α for β,)(for ( ) and $\}\{$ for $\{\}$. Substituting therefore those forms, and afterwards putting $(\,)^s \times \frac{1}{x^{rs}}$ for $)(^s$, and $\{\}^s \times \frac{1}{x^{rs}}$ for $\}\{^s$, we find, at last,

$$(6)\ldots\ldots\int (\,)^p \{\}^{n-1} x^{mr-1} dx =$$

$$-\left\{\frac{1}{n\,.\,r\,.\,a} + \frac{\mu r\{\}}{n(n+1)r^2a^2} + \frac{\mu r(\mu r + r)\{\}^2}{n(n+1)(n+2)r^3a^3} + \right\}(\,)^p \{\}^n x^{mr}$$

$$+\left\{\frac{p\alpha}{n(n+1)ra^2} + \frac{2p\alpha\mu r\{\}}{n(n+1)(n+2)r^2a^3} + \frac{3p\,.\,\alpha\mu r(\mu r + r)\{\}^2}{n(n+1)(n+2)(n+3)r^3a^4} + \right\}(\,)^{p-1} \{\}^{n+1} x^{mr}$$

$$-\left\{\frac{p(p-1)\alpha^2}{n(n+1)(n+2)ra^3} + \frac{3p(p-1)\alpha^2\mu r\{\}}{n(n+1)(n+2)(n+3)r^2a^4} + \frac{6p(p-1)\alpha^2\mu r(\mu r + r)\{\}^2}{n(n+1)\ldots\ldots(n+4)r^3a^5} + \right\}(\,)^{p-2} \{\}^{n+2} x^{mr}$$

&c.

+

No. III. *The summation of the series*

$$\frac{1}{1^2 . p} \pm \frac{1}{2^2 . q} + \frac{1}{3^2 . r} \pm \frac{1}{4^2 . s} +, \text{ and}$$

$$\frac{1}{1^2 . p} \times \frac{1}{2} \pm \frac{1}{2^2 . q} \times \left(\frac{1}{2}\right)^2 + \frac{1}{3^2 . r} \times \left(\frac{1}{2}\right)^3 \pm \frac{1}{4^2 . s} \times \left(\frac{1}{2}\right)^4 +,$$ *where* 1, p, q, r, s, *&c. are the figurate numbers of any order.*

The only cases in which these series have hitherto received summation, are those where the figurate numbers are of the first and second order, when they enter into the series $\overset{n}{\underset{m}{Y}}$ of *Spence.*

The series $\frac{1}{1^2} \times \frac{1}{2} + \frac{1}{2^2} \times \left(\frac{1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{1}{2}\right)^3 +$, which is the simplest of the 2nd class, first received summation from *Euler.* We may sum the whole of them in the following manner. If in equation (6) of the former paper, we make $p = 0$, $n - 1 = s$, and, consequently, $\mu = m + s + 1$, there results

$$\int \left\{\;\right\}^{s} x^{mr-1} dx = -\frac{x^{mr}\left\{\;\right\}^{s+1}}{(s+1)r . a} - \frac{\mu r x^{mr}\left\{\;\right\}^{s+2}}{(s+1)(s+2)r^2a^2} - \frac{\mu r(\mu r + r)x^{mr}\left\{\;\right\}^{s+3}}{(s+1)(s+2)(s+3)r^3a^3} - \&c.$$ To make this series terminate, we must observe that $\underset{n+1}{X} . \underset{n+1}{\chi}$ is to be made equal to

$$* - \int \underset{n}{\overset{\prime}{X}} \cdot \underset{n}{\chi}\, dx = \int \frac{\mu r(\mu r + r) \ldots\ldots (\mu r + nr) \left\{\;\right\}^{s+n+1} x^{mr-1} dx}{(s+1)(s+2) \ldots . (s+n+1) r^{n+1} a^{n+1}}.$$

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* Observe that $\underset{\cdot}{X} = \phi(x)$, and $\underset{\cdot}{\chi} = \psi(x)$.

This expression, one of the most useful for the comparison of integrals, may be found in *Simpson*'s Fluxions. By means of it, we get

$$\int \frac{x^{m-1}\,dx}{(1+x^n)^2} = \frac{x^m}{n(1+x^n)} + \frac{m-n}{n}\int\frac{x^{m-1}\,dx}{1+x^n}:$$

$$\int \frac{x^{m-1}\,dx}{(1+x^n)^3} = \frac{x^m}{2n(1+x^n)^2} - \frac{(m-2n)x^m}{1.2.n^2(1+x^n)} + \frac{(m-2n)(m-n)}{1.2.n^2}\int\frac{x^{m-1}\,dx}{1+x^n};$$

$$\int \frac{x^{m-1}\,dx}{(1+x^n)^4} = \frac{x^m}{3n(1+x^n)^3} - \frac{(m-3n)x^m}{2.3.n^2(1+x^n)^2} + \frac{(m-3n)(m-2n)x^m}{1.2.3.n^3(1+x^n)} - \frac{(m-3n)(m-2n)(m-n)}{1.2.3.n^3}\int\frac{x^{m-1}\,dx}{1+x^n};$$

&c. &c.

whence $\int \frac{x^{n-1}\,dx}{(1+x^n)^2} = \frac{x^n}{n(1+x^n)}$, $\int \frac{x^{2n-1}\,dx}{(1+x^n)^3} = \frac{x^{2n}}{2n(1+x^n)^2}$, $\int \frac{x^{3n-1}\,dx}{(1+x^n)^4} = \frac{x^{3n}}{3n(1+x^n)^3}$, &c.

This property was made use of by *John Bernoulli*, for the summation of a particular series (Operum, tom. 4, p. 25); and, by means of it, we may find $\int \frac{x^{n-m-1}\,dx}{1+x^n}$ $\int \frac{x^{m-1}\,dx}{1+x^n}$; for, by the general formula, in the last paper, we easily get

$$\int \frac{x^{m-1}\,dx}{1+x^n} = \frac{x^m}{m(1+x^n)} + \frac{1.nx^{m+n}}{m(m+n)(1+x^n)^2} + \frac{1.2.n^2.x^{m+2n}}{m(m+n)(m+2n)(1+x^n)^3} + \frac{1.2.3.n^3.x^{m+3n}}{m(m+n)(m+2n)(m+3n)(1+x^n)^4} +,$$

Whence, by the property of $\int \frac{x^{rn-1}dx}{(1+x^n)^{r+1}}$, just now given, $\int \frac{x^{n-m-1}dx}{1+x^n} \int \frac{x^{m-1}dx}{1+x^n} =$

$\frac{1}{n,m} \times \frac{x^n}{1+x^n} + \frac{1}{2m(m+n)} \times \frac{x^{2n}}{(1+x^n)^2} + \frac{1\,.\,2\,.\,n}{3m(m+n)(m+2n)} \times \frac{x^{3n}}{(1+x^n)^3} +$, therefore, if $n = 1$, and,

$1, p, q, r, s$, &c. be the figurate numbers of the m^{th} order, $\int \frac{dx}{x^m(1+x)} \int \frac{x^{m-1}dx}{1+x} =$

$\frac{1}{1^2\,.\,p} \times \frac{x}{1+x} + \frac{1}{2^2\,.\,q} \times \frac{x^2}{(1+x)^2} + \frac{1}{3^2\,.\,r} \times \frac{x^3}{(1+x)^3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$

This integral, from $x = 0$, to x infinite, and from $x = 0$, to $x = 1$, gives us the two series

$\frac{1}{1^2\,.\,p} + \frac{1}{2^2\,.\,q} + \frac{1}{3^2\,.\,r} +$, and $\frac{1}{1^2\,.\,p} \times \frac{1}{2} + \frac{1}{2^2\,.\,q} \times \left(\frac{1}{2}\right)^2 + \frac{1}{3^2\,.\,r} \times \left(\frac{1}{2}\right)^3 +$.

First make $m = 1$, whence

$\int \frac{dx\,\mathrm{L}(1+x)}{x(1+x)} = \frac{1}{1^2} \times \frac{x}{1+x} + \frac{1}{2^2} \times \frac{x^2}{(1+x)^2} + \frac{1}{3^2} \times \frac{x^3}{(1+x)^3} +$; Now this series is evidently $=$ $-\overset{2}{\mathrm{L}}\left(1 - \frac{x}{1+x}\right)$; on the other hand, $\int \frac{dx\,\mathrm{L}(1+x)}{x(1+x)}$ found by *Spence*, p. 74, is $\overset{2}{\mathrm{L}}(1+x) - \frac{1}{2}\mathrm{L}^2(1+x)$; so that

(8)…… $\overset{2}{\mathrm{L}}(1+x) - \frac{1}{2}\mathrm{L}^2(1+x) = -\overset{2}{\mathrm{L}}\left(1 - \frac{x}{1+x}\right)$; which is, in fact, the elegant theorem given by *Mr. Herschel*, in the last No. of the Repository.

Under the form it here assumes, it is of use to find the value of the first member when x is infinite; which value is $-\overset{2}{\mathrm{L}}(0)$. The absolute necessity of such a theorem, will presently appear.

Make $m = 2$, in equation (7): Then $\int \frac{dx}{x^2(1+x)} \int \frac{xdx}{(1+x)} =$

$$\frac{1}{1^2 . 2} \times \frac{x}{1+x} + \frac{1}{2^2 . 3} + \frac{x^2}{(1+x)^2} + \frac{1}{3^2 . 4} \times \frac{x^3}{(1+x)^3} + = \overset{2}{\mathrm{L}}(1+x) - \frac{1}{2} . \mathrm{L}^2(1+x) + \frac{\mathrm{L}(1+x)}{x} - 1,$$ by integrating and correcting. When x is infinite, we have, by equation (8),

$$\frac{1}{1^2 . 2} + \frac{1}{2^2 . 3} + \frac{1}{3^2 . 4} + = -\overset{2}{\mathrm{L}}(0) - 1,$$ and, when $x = 1$,

$$\frac{1}{1^2 . 2} \times \frac{1}{2} + \frac{1}{2^2 . 3} \times \left(\frac{1}{2}\right)^2 \times \frac{1}{3^2 . 4} \times \left(\frac{1}{2}\right)^3 + = \overset{2}{\mathrm{L}}(2) - \frac{1}{2}\mathrm{L}^2(2) + \mathrm{L} . 2 - 1;$$ both

of which enter in *Spence*'s $\overset{n}{\underset{m}{\mathrm{Y}}}$, and agree with the results found by his formula, p. 117.

Next make $m = 3$, and we have to find $\int \frac{dx}{x^3(1+x)} \int \frac{x^2dx}{1+x}$; This, being integrated and corrected, gives

$$\frac{1}{1^2 . 3} \times \frac{x}{1+x} + \frac{1}{2^2 . 6} \times \frac{x^2}{(1+x)^2} + \frac{1}{3^2 . 10} \times \frac{x^3}{(1+x)^3} + = \frac{1}{2x} - \frac{\mathrm{L}(1+x)}{2x^2} + \frac{\mathrm{L}(1+x)}{x} + \overset{2}{\mathrm{L}}(1+x) - \frac{1}{2}\mathrm{L}^2(1+x) - \frac{5}{4}.$$ whence

$$\frac{1}{1^2 . 3} + \frac{1}{2^2 . 6} + \frac{1}{3^2 . 10} + \frac{1}{4^2 . 15} + = -\overset{2}{\mathrm{L}}(0) - \frac{5}{4},$$ and

$$\frac{1}{1^2\cdot3}\times\frac{1}{2}+\frac{1}{2^2\cdot6}\times\left(\frac{1}{2}\right)^2+\frac{1}{3^2\cdot10}\times\left(\frac{1}{2}\right)^3+=\overset{2}{L}(2)-\frac{1}{2}L^2(2)+\frac{1}{2}L(2)-\frac{3}{4}.$$

In like manner, we may proceed for higher values of m, the integrations, which depend on *Spence*, p. 74, having no difficulty.

We shall find that the series $\frac{1}{1^2\cdot p}+\frac{1}{2^2\cdot q}+\frac{1}{3^2\cdot r}+$ &c., is always given by $\overset{2}{L}(0)$, without logarithms.

If, in the sums thus found, we put $-x$ for x, and change the sign of every term, we get the sum of the series $\frac{1}{1^2\cdot p}\times\frac{x}{1-x}-\frac{1}{2^2\cdot q}\times\frac{x^2}{(1-x)^2}+$ where $\frac{x}{1-x}$ may equal any positive number z, which will give $x=\frac{z}{1+z}$.

Thus, we have

$$\frac{1}{1^2\cdot2}\times\frac{x}{1-x}-\frac{1}{2^2\cdot3}\times\frac{x^2}{(1-x)^2}+\frac{1}{3^2\cdot4}\times\frac{x^3}{(1-x)^3}-=-\overset{2}{L}(1-x)+\frac{1}{2}L^2(1-x)+\frac{L(1-x)}{x}+1,$$

$=$ (by *Mr. Herschel's* theorem) $\overset{2}{L}\left(\frac{1}{1-x}\right)+\frac{L(1-x)}{x}+1$. If $z=1$, $x=\frac{1}{2}$, and

$$\frac{1}{1^2\cdot2}-\frac{1}{2^2\cdot3}+\frac{1}{3^2\cdot4}-=\overset{2}{L}(2)-2L\cdot2+1;$$ and if $z=\frac{1}{2}$, $x=\frac{1}{3}$, and

$\frac{1}{1^2\cdot2}\times\frac{1}{2}-\frac{1}{2^2\cdot3}\times\left(\frac{1}{2}\right)^2+\frac{1}{3^2\cdot4}\times\left(\frac{1}{2}\right)^3-=\overset{2}{L}\left(\frac{3}{2}\right)-3L\left(\frac{3}{2}\right)+1$. I have purposely taken these examples, because they enter into *Mr. Spence's* series, and may be compared with the results derived from his formula. It may be worth while to notice that

$$\frac{1}{1^n \cdot p} \times \frac{x}{1+x} + \frac{1}{2^n \cdot q} \times \frac{x^2}{(1+x)^2} + \frac{1}{3^n \cdot r} \times \frac{x^3}{(1+x)^3} + =$$

$$(n) \ldots\ldots \int \frac{dx}{x(1+x)} \int \frac{dx}{x(1+x)} \int \frac{dx}{x^m(1+x)} \int \frac{x^{m-1}\,dx}{1+x},$$

If the classes of series we have considered were likely to be of much use, it would be better to find at once the general integral $\int \frac{dx}{x^m(1+x)} \int \frac{x^{m-1}\,dx}{1+x}$; but as this does not appear to be the case, I shall not encumber the pages of the Repository with any further discussion on the subject.*

No. IV. *To express* $x^n \pm \frac{1}{x^n}$, *n being a whole positive number, by series arranged according to the powers of* $x + \frac{1}{x}$.

$$\left(x + \frac{1}{x}\right)^n = x^n + nx^{n-2} + \frac{n(n-1)}{2} x^{n-4} + \ldots\ldots + \frac{n(n-1)}{2} \times \frac{1}{x^{n-4}} + n \times \frac{1}{x^{n-2}} + \frac{1}{x^n} =$$

$$\left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-}}\right) + \frac{n(n-1)}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.$$

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* Note. When I remarked the agreement of Equation 8, with Mr. *Herschel's* Theorem, it did not occur to me, that it had been given under the form it here assumes (with the exception of the notation) by Mr. *Ivory* : *Mathematical Repository, vol.* 2, *page* 134.

By changing n into $n-2$ we get $n(x+\frac{1}{x})^{n-2} = n(x^{n-2} + \frac{1}{x^{n-2}}) + n(n-2)\,(x^{n-4} + \frac{1}{x^{n-4}})$ +, and subtracting this from the former equation there results

$$(x+\frac{1}{x})^{n} - n(x+\frac{1}{x})^{n-2} = (x^{n} + \frac{1}{x^{n}}) + * + \left(\frac{n(n-1)}{2} - \frac{2n(n-2)}{2}\right)(x^{n-4} + \frac{1}{x^{n-4}}) + \&c.$$

we might next eliminate $x^{n-4} + \frac{1}{x^{n-4}}$, and so on. But, by this mode of reversing the series, we should not easily perceive the law of the coefficients. The general form of the series is, however, plain enough; viz.

$$x^{n} + \frac{1}{x^{n}} = (x+\frac{1}{x})^{n} + \overset{\prime\prime}{R}(x+\frac{1}{x})^{n-2} + \overset{\prime\prime\prime\prime}{R}(x+\frac{1}{x})^{n-4} + \ldots\ldots \overset{\prime\prime\ldots 2r}{R}(x+\frac{1}{x})^{n-2r} + \ldots\ldots\ldots (1);$$

and the coefficients may be found, by a property of the funcion we are to expand, viz.

$$\frac{d\,.\,(x^{n} \pm \frac{1}{x^{n}})}{dx} \times \frac{x}{n} = x^{n} \mp \frac{1}{x^{n}};$$

by means of which the function may be reproduced in another form. If we take the differential of (1), and divide by $\frac{dx}{x}$, we find

$$n(x^{n} - \frac{1}{x^{n}}) = \left\{ n(x+\frac{1}{x})^{n-1} + (n-2)\overset{\prime\prime}{R}(x+\frac{1}{x})^{n-3} + (n-4)\overset{\prime\prime\prime\prime}{R}(x+\frac{1}{x})^{n-5} + \right\}(x-\frac{1}{x}) \ldots\ldots\ldots (2).$$

Take the differential again, and divide as before; there results

$$n^2\left(x^n + \frac{1}{x^n}\right) = \Big\{ n(n-1)\left(x + \frac{1}{x}\right)^{n-2} + (n-2)(n-3)\overset{\prime\prime}{\mathrm{R}}\left(x + \frac{1}{x}\right)^{n-4} + (n-4)(n-5)\overset{\prime\prime\prime\prime}{\mathrm{R}}\left(x + \frac{1}{x}\right)^{n-6} + \Big\}\left(x - \frac{1}{x}\right)^2$$

$$+ \Big\{ n\left(x + \frac{1}{x}\right)^{n-1} + (n-2)\overset{\prime\prime}{\mathrm{R}}\left(x + \frac{1}{x}\right)^{n-3} + (n-4)\overset{\prime\prime\prime\prime}{\mathrm{R}}\left(x + \frac{1}{x}\right)^{n-5} + \Big\}\left(x + \frac{1}{x}\right).$$

But $\left(x - \frac{1}{x}\right)^2 = \left(x + \frac{1}{x}\right)^2 - 4$, by the substitution of which, our last equation becomes

$$n^2\left(x^n + \frac{1}{x^n}\right) = (n + n(n-1))\left(x + \frac{1}{x}\right)^n + \left((n-2)\overset{\prime\prime}{\mathrm{R}} + (n-2)(n-3)\overset{\prime\prime}{\mathrm{R}} - 4n(n-1)\right)\left(x + \frac{1}{x}\right)^{n-2}$$

$$+ \left((n-4)\overset{\prime\prime\prime\prime}{\mathrm{R}} + (n-4)(n-5)\overset{\prime\prime\prime\prime}{\mathrm{R}} - 4(n-2)(n-3)\overset{\prime\prime}{\mathrm{R}}\right)\left(x + \frac{1}{x}\right)^{n-4}$$

$$+ \left((n-6)\overset{\prime\prime\prime\prime\prime\prime}{\mathrm{R}} + (n-6)(n-7)\overset{\prime\prime\prime\prime\prime\prime}{\mathrm{R}} - 4(n-4)(n-5)\overset{\prime\prime\prime\prime}{\mathrm{R}}\right)\left(x + \frac{1}{x}\right)^{n-6} +$$

&c.

or, because $n - r + (n-r)(n-r-1) = (n-r)^2$

$$n^2\left(x^n + \frac{1}{x^n}\right) = n^2\left(x + \frac{1}{x}\right)^n + \left((n-2)^2\overset{\prime\prime}{\mathrm{R}} - 4n(n-1)\right)\left(x + \frac{1}{x}\right)^{n-2}$$

$$+ \left((n-4)^2\overset{\prime\prime\prime\prime}{\mathrm{R}} - 4(n-2)(n-3)\overset{\prime\prime}{\mathrm{R}}\right)\left(x + \frac{1}{x}\right)^{n-4}$$

……………………………………………………

$$+\left((n-2r)^2\overset{\prime\prime\cdots\cdot 2r}{\mathrm{R}}-4(n-2r+2)(n-2r+1)\overset{\prime\prime\cdots\cdot(2r-2)}{\mathrm{R}}\right)\left(x+\frac{1}{x}\right)^{n-2r}.$$

&c.

If we now multiply equation (1) by n^2, and compare it with this, we find

$$n^2\overset{\prime\prime\cdots\cdot 2r}{\mathrm{R}}=(n-2r)^2\overset{\prime\prime\cdot\cdot 2r}{\mathrm{R}}-4(n-2r+2)(n-2r+1)\overset{\prime\prime\cdots\cdot(2r-2)}{\mathrm{R}},\ \text{whence}\ \overset{\prime\prime\cdots\cdot 2r}{\mathrm{R}}=-\frac{(n-2r+2)(n-2r+1)}{r(n-r)}\overset{\prime\prime\cdots\cdot(2r-2)}{\mathrm{R}};$$

whence, making r successively 1, 2, 3, &c., and observing that $\overset{0}{\mathrm{R}}=1$,

$$\overset{\prime\prime}{\mathrm{R}}=-n,\ \overset{\prime\prime\prime\prime}{\mathrm{R}}=-\frac{n-3}{2}\overset{\prime\prime}{\mathrm{R}},\ \overset{\prime\prime\cdots\cdot 6}{\mathrm{R}}=-\frac{(n-4)(n-5)}{3(n-3)}\overset{\prime\prime\prime\prime}{\mathrm{R}},\ \overset{\prime\prime\prime\cdots\cdot 8}{\mathrm{R}}=-\frac{(n-6)(n-7)}{4(n-4)}\overset{\prime\prime\cdots\cdot 6}{\mathrm{R}},\ \text{\&c. or}$$

$$\overset{\prime\prime}{\mathrm{R}}=-n,\ \overset{\prime\prime\prime\prime}{\mathrm{R}}=\frac{n(n-3)}{2},\ \overset{\prime\prime\cdots\cdot 6}{\mathrm{R}}=-\frac{n(n-4)(n-5)}{2\cdot 3},\ \overset{\prime\prime\cdots\cdot 8}{\mathrm{R}}=\frac{n(n-5)(n-6)(n-7)}{2\cdot 3\cdot 4},\ \text{\&c. and}$$

$$x^n+\frac{1}{x^n}=\left(x+\frac{1}{x}\right)^n-n\left(x+\frac{1}{x}\right)^{n-2}+\frac{n(n-3)}{2}\left(x+\frac{1}{x}\right)^{n-4}-\frac{n(n-4)(n-5)}{2\cdot 3}\left(x+\frac{1}{x}\right)^{n-6}+,$$

and equation (2) gives us

$$x^n-\frac{1}{x^n}=\left\{\left(x+\frac{1}{x}\right)^{n-1}-\frac{n-2}{1}\left(x+\frac{1}{x}\right)^{n-3}+\frac{(n-3)(n-4)}{2}\left(x+\frac{1}{x}\right)^{n-5}-\right\}\left\{\left(x+\frac{1}{x}\right)^2-4\right\}^{\frac{1}{2}}.$$

ARTICLE XXI.

Certain properties appertaining to a Triangle inscribed in a given Circle, having two of its sides in a given ratio.

By Mr. Cunliffe, *R. M. College.*

ACRBM, and CNL, are two given circles, touching each other inwardly at C; CL, CM, the diameters passing through the point of contact C; AQNR, is a chord of the greater circle, cutting CM, within the less circle, at right angles in Q, and the circumference of the less circle in N. Draw the lines CA, CR, CNB, AB, AM; also draw the line NLP meeting AM in P, and join CP.

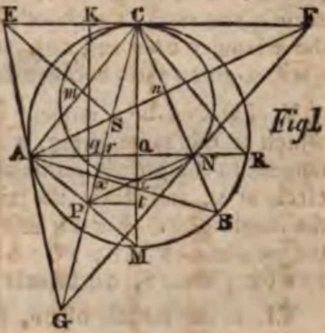


Fig 1

I. The triangles ACN and BCA are similar. For ∠ CAR = ∠ CRA = ∠ CBA; and ∠ ACN (ACB) is common to both the triangles; wherefore the triangle ACN is similar to the triangle BCA, and consequently AC : CN :: BC : AC.

II. The sides AC and CN, or BC and CA have a given ratio. For by obvious properties $AC^2 = CM \times CQ$, and $CN^2 = CL \times CQ$; therefore $AC^2 : CN^2 :: CM : CL :: BC^2 : CA^2$, a given raratio, because CM and CL are both given; wherefore the ratio of AC to CN, or of BC to AC is given.

III. CP is the diameter of a circle passing through the points A, C, N, P, and is given.

For ∠ CAM (CAP) being in a semi-circle is a right angle; as also ∠ CNL (CNP), therefore CP is the diameter of a circle passing through the points A, C, N, P. Now ∠ CAN (CAQ) = ∠ CPN, and ∠ CAQ = ∠ CMA, wherefore the right angled triangles CNP and CAM are similar, and therefore $AC^2 = CM \times CQ : CN^2 = CL \times CQ :: CM : CL :: CM^2 : CP^2 = CL \times CM$; hence it appears, that CP is a mean proportional between CL and CM, which are both given, consequently CP is given also. Or we may prove that CP is given as follows. ∠ CAN (CAQ) = ∠ CPN (CPL); and ∠ CAQ = ∠ CMA (CMP), wherefore the triangles CMP, and CPL are similar, therefore CL : CP :: CP : CM, that is CP is a mean proportional between the given lines

CL and CM, and is therefore given; consequently the locus of P is a given circle whose centre is C.

IV. The triangle LMP is similar to the triangle ACN or BCA and therefore the sides LP and MP have a given ratio.

For ∠ ACN (ACB) is the supplement of ∠ APN; also ∠ MPL is the supplement of the same ∠ APN; wherefore the ∠ ACB = ∠ MPL: also angle CBA = ∠ CMA (LMP), and therefore the triangles ACB and LPM are similar, and as the sides AC and BC, have been shewn to have a given ratio to each other; therefore the sides, LP and MP, of the triangle LPM, must have the same given ratio to each other. Or we have proved (III.), that the triangles CMP and CPL are similar, that is CL : CP :: LP : MP, a given ratio because CL and CP are both given.

V. Let ECF be a tangent to both the given circles at C, or which is the same thing, a line perpendicular to CM; GAE a tangent to the greater circle at A, and GNF a tangent to the less circle at N, and draw the right line GC; then GC will bisect the angle EGF. For FN = FC, and EA = EC; and because of the parallels AN, EF; FG : GE :: FN = FC : EA = EC, wherefore by 3 Euc. 6, GC bisects the angle EGF.

VI. S, the middle of CP, is the centre of the inscribed circle, of the triangle EGF.

Draw AS and ES; then in the triangles ECS and EAS, CS = AS, EC = EA, and ES is common, therefore the triangles ECS and EAS are equal in all respects, and consequently ∠ AES = ∠ CES, that is the angle CEG is bisected by ES.

And drawing FS we may in the same manner prove that the ∠ CFG is bisected by FS; wherefore S is the centre of the inscribed circle of the triangle EGF.

VII. CS, the distance of the centre of the inscribed circle of the triangle EGF, from the point C, is given. For CS is the half of CP, which has been shewn to be given.

VIII. The points C, P and G are in the same right line. We have proved that GC bisects the angle EGP, (V.), and it is well known that the lines bisecting the angles of a plane triangle intersect in the same point; consequently CG must pass through S; but S is the middle of CP, therefore GC must pass through P.

IX. Draw PK perpendicular to EF, cutting AN in q; also draw Pt perpendicular to CM: then it is evident that $\mathrm{P}q = \mathrm{Q}t$, and $q\mathrm{K} = \mathrm{QC}$; and the circle whose diameter is CP, will obviously pass through the points C, N, t, P, A, K. By the property of the circle $\mathrm{A}q \times q\mathrm{N} = \mathrm{P}q \times q\mathrm{K} = \mathrm{Q}t \times \mathrm{QC}$; also $\mathrm{AQ} \times \mathrm{QN} = \mathrm{Q}t \times \mathrm{QC}$ and therefore $\mathrm{A}q \times q\mathrm{N} = \mathrm{AQ} \times \mathrm{QN}$, wherefore $q\mathrm{N} = \mathrm{AQ}$, $\mathrm{A}q = \mathrm{QN}$. For if the same right line be cut at two different points, into two parts, whose rectangles are equal to each other; then the two greater segments in each case will be equal, and consequently, the two less segments will be equal also.

X. $QL \times QM = Qt^2$, and therefore Qt is equal to a tangent from Q, to the circle whose diameter is LM.

Because of the parallels Pq; QM.

$AQ : Aq = QN :: QM : Pq = Qt$; also $Nq = AQ : QN :: Pq = Qt : QL$, wherefore $QM : t :: Qt : QL$, and hence $QL \times QM = Qt^2$; therefore Qt, is equal to a tangent from Q, to the circle whose diameter is LM, as is very well known.

XI. AB cuts CP at right angles.

Let x, be the point of intersection of AB and CP: then $\angle APC\ (APx) = \angle ANC\ (QNC)$; and $\angle MAB\ (PAx) = \angle MCN\ (QCN)$: hence $\angle APx + \angle PAx = \angle QNC + \angle QCN$; but $\angle QNC + \angle QCN$ = a right angle, and consequently $\angle APx + \angle PAx$ = a right angle, and therefore $\angle PxA$ is a right angle.

XII. Imagine the sides EG and FG, of the triangle EGF, to be bisected at the points of contact A and N, of the given circles: then it is well known that the triangle EGF will be the least that can be so described. (See schol. to theorem 8, Simpson on the Maxima and Minima.)

It is moreover well known that the triangle ACN (see the said schol.) will be the greatest that can be inscribed in the triangle EGF, and consequently the triangle ACN will be the greatest that can be described to have its vertex at C, and its base AN at right angles to CM; and the extremities of the base A and N in the circumferences of the two given circles. And furthermore the triangle ACN will be inscribed in the circle whose diameter is CP.

XIII. Let r, be the intersection of CP and AN; then $GA = AE = EC$, or $GE = 2EC$: and because ES bisects the angle CEG, $EC : GE = 2EC :: 1 : 2 :: CS : 2CS = SG = CP$, therefore

$GC = \dfrac{3CP}{2}$; but GC is bisected in r, therefore $Cr = \dfrac{3CP}{4}$; and

hence $rP = \dfrac{CP}{4}$. Moreover because of the parallels Pt and

rQ, $Cr = \dfrac{3CP}{4} : rP = \dfrac{CP}{4} :: 3 : 1 :: CQ : Qt$.

In article (XII.) we have made out the following remarkable property of the greatest triangle that can be inscribed in a given circle, having two of its sides in a given ratio; viz. The base of the greatest triangle that can be inscribed in a given circle, having the other two sides in a given ratio, cuts the diameter passing through the vertex, at the distance of $\frac{1}{4}$ of the diameter from the vertex.

We shall now go on to determine the greatest triangle that can be inscribed in a given circle, having two of its sides in a given ratio; and in order to proceed clearly, it appears necessary to put down the following problem with its construction, viz.

XIV. LTM (Fig. 2.) is a given circle, and C a given point without the circle; it is required to find a point Q, in the line CLM drawn through the centre O, from whence a tangent QT being drawn, QT may be to QC in the given ratio of m to n.

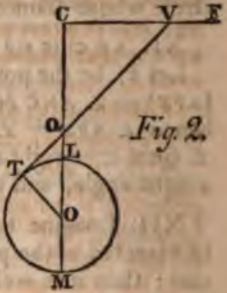


Fig. 2.

Draw CF perpendicular to CO, in which take CV a fourth proportional to m, n and $LO = \frac{1}{2}LM$: then CV will be given because LO is given, and therefore V becomes a given point. From V, draw the tangent VT to the circle, cutting CO in Q, and the thing is done.

DEMONSTRATION. Draw the radius OT, to the point of contact T, then the right angled triangles QTO and QCV being similar, $QT : QC :: OT = OL : CV$; but by the construction $m : n :: OL = OT : CV$. Consequently $QT : QC :: m : n$, the given ratio.

XV. In a given circle to inscribe the greatest triangle possible having two of its sides in a given ratio.

Draw a diameter CM (*Fig* 3.) upon which take CL, so that CM shall be to CL, in the duplicate of the given ratio of the sides. Upon CL and LM as diameters, describe two other circles, which will touch each other in L. Draw CF perpendicular to CM, in which take $CV = 3OL = \frac{3LM}{2}$: from V draw a tangent VT, to the circle whose diameter is LM, cutting CL in Q. Through Q draw AQN at right angles to CM, cutting the given circle in A, and the circle whose diameter is CL in N. Draw AM and NL to meet AM in P; draw CP and CN; produce CN till it meets the given circle in B and join AB; then ACB is the greatest triangle that can be inscribed in the given circle,

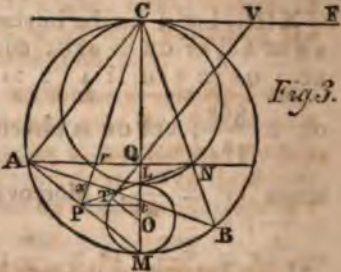


Fig. 3.

so that $BC^2 : AC^2 :: CM : CL$ or $BC : AC :: CM : CP$, CP being a mean proportional between CL and CM.

Draw P*t* parallel to AN, meeting LM in *t*: by the construction $CV : OL = OT :: 3 : 1$, and per similar right-angled triangles $CV : OT = OL :: CQ : QT = Qt$. Also by reason of the parallels AQ, P*t*, $CQ : Qt :: Cr : rP$; consequently $CV : OL :: Cr : rP :: 3 : 1$; wherefore (Art. XII.) the triangle ACN, is the greatest that can possibly be inscribed in the circle whose diameter is CP, having the sides AC, CN in the given ratio of CM to CP: and therefore the triangle BCA being similar to ACN, will be the greatest that can be inscribed in the given circle, having the sides BC, AC in the same given ratio of CM to CP.

That the triangle ACB is a maximum when ACN is a maximum may be shewn as follows:

Area of the $\triangle$ ACN : area Δ ACB $:: CN : CB :: CL : CM$ a given ratio, because CL and CM are both given. Therefore when the triangle ACN is a maximum the triangle ACB must be a maximum also.

XVI. In a given sphere it is required to inscribe the greatest oblique cone possible, having its greatest and least slant sides in a given ratio.

Let AN (*Fig.* 1), represent the diameter of the base of the greatest oblique cone, that can be inscribed in the sphere whose diameter is CP, and having the greatest and least slant sides, CA and CN, in the given ratio of CM to CP.

From the Schol. to Theorem 19, Simpson on the maxima and minima, we gather that in the case of the greatest inscribed cone ACN, $Gr = 2Cr$, or $Cr = \frac{1}{3}GC$, and therefore by reason of the parallels A*r*, EC; $GA = 2AE = 2EC$; and hence $EG = 3AE = 3EC$.

Now ES bisects $\angle$ GEC, therefore $GE + EC = 4\,EC : GC :: EC : CS = \frac{GC}{4}$; and $CP = 2CS = \frac{GC}{2}$, and hence $GC = 2CP$. Therefore $rP = CP - Cr = \frac{GC}{2} - \frac{GC}{3} = \frac{GC}{6} = \frac{2CP}{6} = \frac{CP}{3}$; $Cr = \frac{GC}{3} = \frac{2CP}{3}$.

Then because of the parallels *r*Q, P*t*, and from what has been deduced, $CQ : Qt :: Cr = \frac{2CP}{3} : rP = \frac{CP}{3} :: 2 : 1$.

From which conclusion we derive the following method of constructing the problem.

XVII. Upon CF (*Fig.* 3.) take CV = 2OL = LM, and draw the tangent VT cutting CL in Q: through Q, draw AQN at right angles to CL, meeting the circle whose diameter is CM in A, and the circle whose diameter is CL in N. Then in the triangle ACN, the base AN is the diameter of the base of the greatest oblique cone, that can be inscribed in the sphere whose diameter is CP. And AC, NC are the greatest and least slant sides of the cone, and are in the given ratio of CM to CP.

DEMONSTRATION. The right angled triangles VCQ and OTQ are similar, therefore CQ : QT = Q*t* :: CV = 2OL = 2OT : OT :: 2 : 1. And by reason of the parallels *r*Q, P*t*; CQ : Q*t* :: C*r* : *r*P :: 2 : 1; therefore (Art. XVI.) in the triangle ACN the base AN, is the diameter of the base of the greatest oblique cone that can be inscribed in the sphere whose diameter is CP; and AC, NC, the greatest and least slant sides of the cone, are in the given ratio of CM to CP. (Art. XVI.)

Produce CN to meet the circle whose diameter is CM in B, and join AB, MB; then in the triangle ACB the base AB is the diameter of the base of the greatest oblique cone that can be inscribed in the sphere whose diameter is CM, and BC, AC, are the greatest and least slant sides of the cone, and have the given ratio of CM to CP.

That the cone ACB is a maximum, when the cone ACN is a maximum, will appear as follows.

The contents of similar solids are well known to be in the triplicate ratio of their homologous sides; therefore cone ACN : cone ACB :: $AC^3 : BC^3$. And per sim. triangles CAP, CBM, AC : BC :: CP : CM; and therefore cone ACN : cone ACB :: $CP^3 : CM^3$, a given ratio, because CP and CM are both given. Consequently when the cone ACN is a maximum, the cone ACB will be a maximum also.

By help of the foregoing properties, the geometrical construction of a great number of problems may be effected in a manner materially differing from the constructions usually given to those problems. We shall put down a few problems and refer to the properties upon which the constructions are founded.

XVIII. Given the diameter of the circumscribing circle, the difference of the segments of the base made by a perpendicular from the vertical angle, and the ratio of the other two sides to construct the plane triangle.

Let the triangle ACN (*Fig.* 1.) which is circumscribed by the circle whose diameter is CP, represent the required triangle; then CP is given by the question. And A*q* = QN (Art. IX.) therefore *q*Q = P*t* is the difference of the segments of the base AN made by the perpendicular CQ, and is given by the question. Wherefore in the right-angled triangle CP*t*, the hypothenuse CP and leg P*t* being given, the triangle itself will be entirely given.

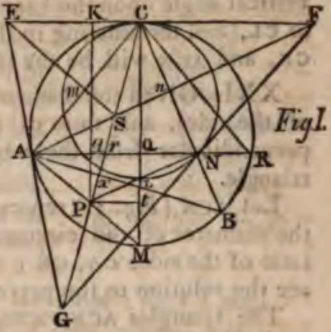


Fig. 1.

The right-angled triangles CAP and CNL are similar, therefore CA : CN :: CP : CL and when the ratio of CA to CN is given, CL becomes known; whence the following construction is derived.

Form the right-angled triangle C*t*P; and in C*t* take CL such, that CP may be to CL in the given ratio of the sides; upon CL describe a semicircle, and through P and L draw PLN meeting its circumference in N; draw NQ perpendicular to CL, meeting the given circle in A: and ACN will be the required triangle, as is evident from the analysis.

XIX. Given the diameter of the circumscribing circle, the difference of the angles at the base, and the ratio of the other two sides, to construct the plane triangle.

(*Fig.* 1.) ∠CNA = CAB = (CA*x*); therefore ∠CNA − ∠CAN = ∠ CA*x* — ∠ CA*r* = ∠ *r*A*x*. Also ∠ A*rx* = ∠C*r*Q; and the angles *rx*A, *r*QC are right angles (Art XI.); therefore ∠*r*A*x* = ∠ *r*CQ (PC*t*); and hence ∠ CNA — ∠ CAN = PC*t*, the difference of the angles at the base, which is given by the question. Wherefore in the right-angled triangle C*t*P the hypothenuse CP, and angle PC*t*, being given, the triangle PC*t* becomes entirely known; consequently the problem is now reduced to the preceding problem, and may therefore be constructed in the same manner.

XX. Given the diameter of the circumscribing circle, perpendicular from the vertical angle upon the base, and the ratio of the sides, to construct the plane triangle.

Let ACN (*Fig.* 1.) represent the required triangle; then CP, the diameter of the circumscribing circle, is given by the question. And per sim. triangles CAP, CNL; AC : CN :: CP : CL, a given ratio, wherefore CL becomes known. Also CP : CL :: CM : CP, and hence CM becomes known; and the construction will be as follows: Upon CM and CL, describe two semicircles, and upon CL, take CQ, equal to the given perpendicular from the vertical angle upon the base: through Q draw AQN at right angles to CL, and terminating in the semicircles in A and N; join CA, CN, and ACN will be the required triangle.

XXI. Given the diameter of the circumscribing circle, ratio of the sides, and ratio of the segments of the base made by a perpendicular from the vertical angle, to construct the plane triangle.

Let ACN (*Fig.* 1.) represent the required triangle; then CP, the diameter of the circumscribing circle being given, and the ratio of the sides CA, CN; whence CM and CL become known, see the solution to the preceding problem.

The triangles ACN, MPL are similar (Art. IV.) and therefore P*t* divides ML, in the same ratio, as CQ divides AN; that is AQ : QN : : M*t* : *t*L, a given ratio by the question; wherefore ML is divided in a given ratio at *t*, whence N*t* becomes known, and consequently C*t*; and therefore the right-angled triangle CP*t* becomes entirely known. Having formed the triangle CP*t*, the the remainder of the construction will be the same as problem, Art. XVIII.

ARTICLE XXII.

A method of obtaining the sum of an infinite series of the form $Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + Ex^{n+4}$ &c. *in which the quantities* A, B, C, D, &c. *are expounded by a series of the sines or cosines, of circular arcs in arithmetical progression; together with some other curious particulars deduced therefrom.*

By Mr. CUNLIFFE, *R. M. College.*

Before we proceed to the summation of the series, it will be necessary to premise a few properties, of the sines and cosines of circular arcs, for the purpose of explaining the various steps in the course of the subsequent calculations. The properties here alluded to, are contained in the following Lemmata.

LEMMA 1. If there be three circular arcs in arithmetical progression, radius being 1; and if the sine of the mean arc be multiplied by twice the cosine of the common difference, and the sine of either extreme subtracted from the product, the remainder will be the sine of the other extreme.

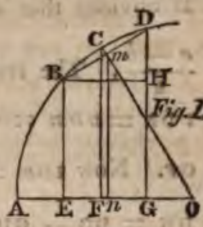


Fig. 1.

Demonstration. Let AB, AC and AD be three arcs of the same circle, in arithmetical progression, arc BC = arc CD, being the common difference. From the centre O, draw the radii OA, OC; the latter of which will bisect the chord BD, at right angles in m, as is very well known. Draw BE, CF, mn, and DG perpendicular to AO; also BH parallel to AO, and meeting DG in H.

By reason of the parallels CF, mn; $\text{OC} : \text{O}m :: \text{CF} : mn$, whence $\text{CF} \times \text{O}m = mn \times \text{OC}$, or $\text{CF} \times 2\text{O}m = 2mn \times \text{OC}$.

And it is very obvious that $mn = \frac{\text{BE} + \text{DG}}{2}$ or $2mn = \text{BE} + \text{DG}$,

therefore $\text{CF} \times 2\text{O}m = 2mn \times \text{OC} = (\text{BE} + \text{DG}) \times \text{OC} = \text{BE} + \text{DG}$, that is, $\text{CF} \times 2\text{O}m = \text{BE} + \text{DG}$ because radius $\text{OC} = 1$.

Now BE is the sine of arc AB; and DG the sine of arc AD; also CF is the sine of the mean arc AC; and Om the cosine of the common difference BC; wherefore from the expression $\text{CF} \times 2\text{O}m = \text{BE} + \text{DG}$ the truth of the lemma is manifest.

Lemma 2. If there be three circular arcs in arithmetical progression, radius being 1; and if the cosine of the mean arc be multiplied by twice the cosine of the common difference, and the cosine of either extreme subtracted from the product, the remainder will be the cosine of the other extreme.

The same preparation and figure will serve here as in the preceding lemma.

Demonstration. By reason of the parallels CF, mn; $\text{OC} : \text{O}m :: \text{OF} : \text{O}n$, whence $\text{O}m \times \text{OF} = \text{O}n \times \text{OC}$, or $2\text{O}m \times \text{OF} = 2\text{O}n \times \text{OC}$. But it is obvious that $\text{O}n = \frac{\text{OE} + \text{OG}}{2}$, or $2\text{O}n = \text{OE} + \text{OG}$; and therefore $2\text{O}m \times \text{OF} = 2\text{O}n \times \text{OC} = (\text{OE} + \text{OG}) \times \text{OC} = \text{OE} + \text{OG}$, because $\text{OC} = 1$; that is, $2\text{O}m \times \text{OF} = \text{OE} + \text{OG}$. Now OE is the cosine of arc AB, and OG is the cosine of arc AD: also OF, is the cosine of the mean arc AC; and Om, the cosine of the common difference, or the cosine of arc BC = arc CD; wherefore the truth of the lemma has been demonstrated.

Lemma 3. If a and b denote two circular arcs, radius 1, of which, a is the greater; then $\sin. a - \sin. b = 2 \sin. \frac{(a-b)}{2} \times \cos. \frac{(a+b)}{2}$.

Demonstration. (*Fig.* 1.) Let arc AD$=a$, arc AB$=b$; then it is obvious that arc AC $= \frac{a+b}{2}$; and arc BC = arc BD = $\frac{a-b}{2}$. The triangles DBH and OFC are similar, therefore OC : DB $=$ 2Dm :: OF : DH, whence DH × OC = DH = 2Dm × OF. Now 2Dm $= 2 \sin. \frac{(a-b)}{2}$; OF $= \cos. \frac{(a+b)}{2}$; and DH = DG − GH = DG − BE $= \sin a - \sin b$: and writing these values in the expression DH = 2Dm × OF, it will become $\sin. a - \sin. b = 2 \sin. \frac{(a-b)}{2} \times \cos. \frac{(a+b)}{2}$. *Q.E.D.*

LEMMA 4. (*Fig.* 1.) Let a and b denote the same things as in the foregoing lemma; then $\cos b - \cos a = 2 \sin \frac{(a-b)}{2} \times \sin \frac{(a+b)}{2}$.

Demonstration. The triangles DBH and OCF being similar; OC : DB = 2Dm :: CF : BH, whence BH × OC = BH = 2Dm × CF. Now CF $= \sin \frac{a+b}{2}$; and BH = EG = OC − OG = $\cos b - \cos a$; the rest as in the preceding lemma; and writing these values in the expression BH = 2Dm × CF, it will become $\cos b - \cos a = 2 \sin \frac{(a-b)}{2} \times \sin \frac{(a+b)}{2}$. *Q.E.D.*

LEMMA 5. If z denotes a circular arc, radius 1; then $1 - \cos z = 2 \times \left(\sin \frac{z}{2}\right)^2$.

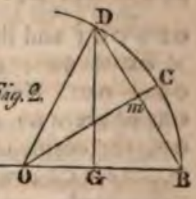


Fig. 2.

Demonstration. Let DB $= z$, be the arc of a circle, radius 1, O its centre; draw the radius OB, and the sine DG perpendicular to OB; also draw the chord BD, and the radius OmC, to bisect the chord BD in m.

Then OC $= \cos z$; and BG = OB − OG $= 1 - \cos z$. Also Bm = Dm $= \sin \frac{z}{2}$. And per similar right-angled triangles BGD, BmO; BO : BD = 2Bm :: Bm : BG, whence BG × BO = BG = 2 × Bm^2; that is, $1 - \cos z = 2 \times \left(\sin \frac{z}{2}\right)^2$. *Q.E.D.*

The preceding properties may be found in most treatises on trigonometry, but it has been deemed expedient to annex them, properly adapted to the subject, with concise demonstrations, for the purpose of rendering what follows more complete.

1. To find the sum of the infinite series $Ax^{n} + Bx^{n+1} + Cx^{n+2} + Dx^{n+3}$ &c. in which the quantities A, B, C, D, &c. denote a series of the sines, of circular arcs in arithmetical progression.

Assume $\dfrac{Ax^{n} - gx^{n+1}}{1 - 2cx + x^{2}} = Ax^{n} + Bx^{n+1} + Cx^{n+2} + Dx^{n+3}$ $+ Ex^{n+4}$ &c.

Multiplying both sides of the expression by $1 - 2cx + x^{2}$, gives $Ax^{n} - gx^{n+1}$

$$\begin{aligned} &= Ax^{n} + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + Ex^{4+n} \text{ \&c.} \\ &\quad - 2cAx^{n+1} - 2cBx^{n+2} - 2cCx^{n+3} - 2cDx^{n+4} \text{ \&c.} \\ &\qquad\qquad + Ax^{n+2} + Bx^{n+3} + Cx^{n+4} \text{ \&c.} \end{aligned}$$

$$\text{or } \left.\begin{array}{l} Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + Ex^{n+4} \text{ \&c.} \\ -2cAx^{n+1} - 2cBx^{n+2} - 2cCx^{n+3} - 2cDx^{n+4} \text{ \&c.} \\ gx^{n+1} + Ax^{n+2} + Bx^{n+3} + Cx^{n+4} \text{ \&c.} \end{array}\right\} = 0$$

By making the coefficients of the like powers of $x = 0$, we have $B = 2cA - g$; $C = 2cB - A$; $D = 2cC - B$; $E = 2cD - C$, &c. &c.

Now let y and z, denote two circular arcs, radius 1; and let c, denote the cosine of z; also let $A = \sin.\ y + (n+1)z$, and $g = \sin.\ y + nz$.

Then from what has been deduced above, and Lemma 1,
$B = 2cA - g = 2c \times \sin y + (n+1)z - \sin y + nz = \sin y + (n+2)z$:
$C = 2cB - A = 2c \times \sin y + (n+2)z - \sin y + (n+1)z = \sin y + (n+3)z$:
$D = 2cC - B = 2c \times \sin y + (n+3)z - \sin y + (n+2)z = \sin y + (n+4)z$, &c:

And these values being written in the assumed expression give

$$\frac{Ax^{n} - gx^{n+1}}{1 - 2cx + x^{2}} = \frac{x^{n} \times \sin.\ y + (n+1)z - x^{n+1} \times \sin.\ y + nz}{1 - 2cx + x^{2}} =$$

$x^{n} \times \sin.\ y + (n+1)z + x^{n+1} \times \sin.\ y + (n+2)z + x^{n+2}$

$\times \sin. y + (n+3)z + x^{n+3} \times \sin. y + (n+4)z$, &c. &c. in infinitum.

(2) Take $n = 0$, and the expression becomes $\frac{\sin y+z - x \sin y}{1-2cx+x^2}$ $= \sin. y+z + x \times \sin. y+2z + x^2 \times \sin. y+3z + x^3 \times \sin. y+4z + x^4 \times \sin. y+5z$ &c. in infinitum. Taking the difference between the two preceding expressions (viz. 2 and 1) there will be had

$$(3)\ \frac{\sin. y+z - x \sin. y + x^{n+1} \times \sin y+nz - x^n \times \sin. y+(n+1)z}{1-2cx+x^2}$$

$= \sin. y+z + x \times \sin. y+2z + x^2 \times \sin. y+3z + x^3 \times \sin. y+4z$ &c. $+ x^{n-1} \times \sin. y+nz$.

Take $x = 1$ and the preceding expression becomes

$$\frac{\sin. y+z - \sin. y + \sin. y+nz - \sin. y+(n+1)z}{2 \times (1-c)}$$

$= \sin. y+z + \sin. y+2z + \sin. y+3z + \sin. y+4z$ &c. $+ \sin. y+nz$. Now, by Lemma 3, $\sin. y+z - \sin. y$ $= 2 \sin. \frac{z}{2} \times \cos. \left(y+\frac{z}{2}\right)$ also $\sin. y+(n+1)z - \sin. y+nz = 2 \sin. \frac{z}{2} \times \cos. y+(2n+1)\frac{z}{2}$, and by means of these

$$\frac{\sin. y+z - \sin. y + \sin. y+nz - \sin. y+(n+1)z}{2 \times (1-c)}$$

$$= \frac{\sin. \frac{z}{2} \times \cos. y+\frac{z}{2} - \sin. \frac{z}{2} \times \cos. y+(2n+1)\frac{z}{2}}{1-\cos z}.$$

Moreover, by Lemma 5, $1 - \cos. z = 2 \left(\sin. \frac{z}{2}\right)^2$;

wherefore $\frac{\sin. y+z - \sin. y + \sin. y+nz - \sin. y+(n+1)z}{2 \times (1-c)} =$

$$= \frac{\sin. \frac{z}{2} \times \cos. y+\frac{z}{2} - \sin. \frac{z}{2} \times \cos. y+(2n+1)\frac{z}{2}}{1-\cos. z.} =$$

$$\frac{\sin. \frac{z}{2} \times \cos. \left(y+\frac{z}{2}\right) - \sin. \frac{z}{2} \times \cos. y+(2n+1)\frac{z}{2}}{2\left(\sin. \frac{z}{2}\right)^2} =$$

$\frac{\cos.\ y + \frac{z}{2} - \cos.\ y + (2n+1)\frac{z}{2}}{2\left(\sin.\ \frac{z}{2}\right)}$. Again, by Lemma 4,

$$\cos y + \frac{z}{2} - \cos y + (2n+1)\frac{z}{2} = 2\sin\frac{nz}{2} \times \sin y + (n+1)\frac{z}{2}$$

and hence $\frac{\cos y + \frac{z}{2} - \cos y + (2n+1)\frac{z}{2}}{2\left(\sin\frac{z}{2}\right)} = \frac{\sin\frac{nz}{2} \times \sin y + (n+1)\frac{z}{2}}{\sin\frac{z}{2}}$

$$= \sin y + z + \sin y + 2z + \sin y + 3z + \text{\&c.} + \sin y + nz \qquad (4).$$

To find the sum of the infinite series $Ax^{n} + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + Ex^{n+4}$ &c. in which the quantities A, B, C, D, &c. denote a series of the cosines of circular arcs, in arithmetical progression.

What has been deduced in Art. 1, may, with very little additional trouble, be applied in the present case.

Let y and z, denote two circular arcs, radius 1; and let c denote the cosine of z, as before. Also, put $A = \cos y + (n+1)z$, and $g = \cos y + nz$. Then by Lemma 2, and what has been deduced in Art. 1,

$B = 2cA - g = 2c \times \cos y + (n+1)z - \cos y + nz = \cos y + (n+2)z$. $C = 2cB - A = 2c \times \cos y + (n+2)z - \cos y + (n+1)z = \cos y + (n+3)z$. $D = 2cC - B = 2c \times \cos y + (n+3)z - \cos y + (n+2)z = \cos y + (n+4)z$. $E = 2cD - C = 2c \times \cos y + (n+4)z - \cos y + (n+3)z = \cos y + (n+5)z$, &c. &c.

(5). And therefore,

$$\frac{Ax^{n} - gx^{n+1}}{1 - 2cx + x^{2}} = \frac{x^{n} \times \cos y + (n+1)z - x^{n+1} \times \cos y + nz}{1 - 2cx + x^{2}}$$

$= x^{n} \times \cos y + (n+1)z + x^{n+1} \times \cos y + (n+2)z + x^{n+2} \times \cos y + (n+3)z + x^{n+3} \times \cos y + (n+4)z + x^{n+4} \times \cos y + (n+5)z$ &c. in infinitum.

(6). Take $n = 0$; and the last expression becomes

$\frac{\cos y + z - x \times \cos y}{1 - 2cx + x^{2}} = \cos y + z + x \times \cos y + 2z + x^{2} \times \cos y + 3z + x^{3} \times \cos y + 4z + x^{4} \times \cos y + 5z$ &c. in infinitum.

And taking the difference between the two preceding expressions, (viz. 6 and 5) there will be had

$$\frac{\cos y+z-x\times\cos y+x^{n+1}\times\cos y+nz-x^{n}\times\cos y+(n+1)z}{1-2cx+x^{2}}$$

$= \cos y+z+x\times\cos y+2z+x^{2}\times\cos y+3z+x^{3}\times\cos y+4z$
$+\,x^{4}\times\cos y+5z$ &c. $+\,x^{n-1}\times\cos y+nz$.

(7). Take $x=1$, and the last expression will become

$$\frac{\cos y+z-\cos y+\cos y+nz-\cos y+(n+1)z}{2\times(1-c)}$$

$$=\frac{\cos y+z-\cos y+\cos y+nz-\cos y+(n+1)z}{2\times(1-\cos z)}$$

$=\cos y+z+\cos y+2z+\cos y+3z+\cos y+4z+\cos y+5z$ &c.
$+\cos y+nz$.

Now, by Lemma 4, $\cos y-\cos y+z=2\sin\frac{z}{2}\times\sin y+\frac{z}{2}$;

and $\cos y+nz-\cos y+(n+1)z=2\sin\frac{z}{2}\times\sin y+(2n+1)\frac{z}{2}$.

Also, by Lemma 5, $1-\cos z=2\left(\sin\frac{z}{2}\right)^{2}$, and therefore

$$\frac{\cos y+z-\cos y+\cos y+nz-\cos y+(n+1)z}{2\times(1-\cos z)}$$

$$=\frac{\sin\frac{z}{2}\times\sin y+(2n+1)\frac{z}{2}-\sin\frac{z}{2}\times\sin y+\frac{z}{2}}{2\times\left(\sin\frac{z}{2}\right)^{2}}$$

$$=\frac{\sin y+(2n+1)\frac{z}{2}-\sin y+\frac{z}{2}}{2\times\sin\frac{z}{2}}=\cos y+z+\cos y+2z$$

$+\cos y+3z+\cos y+4z$ &c. $+\cos y+nz$.

But, by Lemma 3, $\sin y+(2n+1)\frac{z}{2}-\sin y+\frac{z}{2}=2\sin\frac{nz}{2}$
$\times\cos y+(n+1)\frac{z}{2}$.

And therefore,

$$\frac{\sin y+(2n+1)\frac{z}{2}-\sin y+\frac{z}{2}}{2\sin\frac{z}{2}}=\frac{\sin\frac{nz}{2}\times\cos y+(n+1)\frac{z}{2}}{\sin\frac{z}{2}}$$

$=\cos y+z+\cos y+2z+\cos y+3z+\cos y+4z$&c.$+\cos y+nz$.

By dividing the expression for the sum of the sines, obtained in Art. 4, by the expression for the sum of the cosines just deduced, we shall have the following remarkable formula or theorem;

$$\frac{\sin y + (n+1)\frac{z}{2}}{\cos y + (n+1)\frac{z}{2}} = \text{tangent } y + (n+1)\frac{z}{2} =$$

$$\frac{\sin y+z + \sin y+2z + \sin y+3z + \&c. + \sin y+nz}{\cos y+z + \cos y+2z + \cos y+3z + \&c. + \cos y+nz}.$$

ARTICLE XXIII.

On the popular methods of approximation.

By Mr. W. G. HORNER, *Bath.*

If the following concise investigation, and the subsequent remarks, shall be so fortunate as to induce a more correct estimate of the utility of these valuable theorems, than seems at present to prevail, even among respectable mathematicians, the wish of the writer will be amply satisfied.

Let $\phi x = x^{n} + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots p_m x = p_n \ldots\ldots$ [1]

represent any equation whatever, finite or infinite; and suppose that, on making $x = \text{R} + z$, the equation transforms to

$$\Delta = az + bz^2 + cz^3 + \ldots\ldots z^{n} \qquad [2]$$

Then, it is manifest that

$$\Delta = p_n - (\text{R}^{n} + p_1\text{R}^{n-1} + p_2\text{R}^{n-2} + p_3\text{R}^{n-3} \ldots) = p - \phi\text{R}$$

$$a = n\text{R}^{n-1} + n-1 .\, p_1\text{R}^{n-2} + n-2 .\, p_2\text{R}^{n-3} + \ldots = \left(\frac{d\phi\text{R}}{d\text{R}} =\right)\text{D}\phi\text{R}$$

$$b = \frac{n . n-1}{1 . 2}\text{R}^{n-2} + \frac{n-1 . n-2}{1 . 2} p_1\text{R}^{n-3} + \ldots = \left(\frac{da}{2d\text{R}} =\right)\text{D}^2\phi\text{R}$$

$$c = \frac{n . n-1 . n-1}{1 . 2 . 3}\text{R}^{n-3} + \frac{n-1 . n-2 . n-3}{1 . 2 . 3} p_1\text{R}^{n-4} + \ldots = \left(\frac{db}{3d\text{R}} =\right)\text{D}^3\phi\text{R}$$

and so of the rest; agreeably to TAYLOR's theorem. [3]

Now, if R be so nearly equal to the correct value of x, that the difference z bears but a small ratio to R, the numerical value of Δ, if assumed to be $= az$, will generally agree, to a certain number of figures, with its actual value in equation 2. To that extent, therefore we shall have

$$z = \frac{\Delta}{a} \text{ .... } [4]$$

which is the symbol of NEWTON's and RAPHSON's methods, as well as of the second case of BARLOW's method, first published in No. 12 of this Repository, and since that time in the 8vo edition of Bonnycastle's Algebra.

The effect of equation 4 may very commodiously be augmented by substituting it in the result of the more correct equation $\Delta = az + bz^2$, which thus produces

$$z = \frac{\Delta}{a + bz} = \frac{\Delta}{a + \frac{b\Delta}{a}} = \frac{a\Delta}{a^2 + b\Delta} \qquad [5]$$

equivalent to HALLEY's and De LAGNY's *rational methods.*

If, in these expressions, we make $b = \frac{n - 1 \,.\, a}{2\text{R}}$, we obtain the following commodious formula;

$$z = \frac{\Delta}{a + \frac{(n - 1)\,\Delta}{2\text{R}}} = \frac{2\text{R}\,\Delta}{2a\text{R} + (n - 1)\Delta} \qquad [6]$$

which symbolize the other case of BARLOW's method, or that which he directs us to employ when $\text{R} > 1$. The more terms of the equation that are deficient, the more nearly will the correction thus obtained agree with that deduced from HALLEY's method, with which this is accurately coincident in the case of pure powers. In other cases since it neglects the same quantities cz^3 &c. as Halley's, but on the other hand adopts a quantity,

$$\left(\frac{n-1}{2}\underset{1}{p}\text{R}^{n-3} + \frac{2.n-2}{2}\underset{2}{p}\text{R}^{n-4} + \frac{3.n-3}{2}\underset{3}{p}\text{R}^{n-5} + \,..\frac{n-1.1}{2}\underset{m}{p}\text{R}^{-1}\right)z,$$

which the other does not, it will sometimes be the more accurate method. At the same time, it will be in general much the more convenient, in regard of the smaller numbers which it employs. Adding R, this formula gives

$$(\text{R} + z =)\, x = \frac{2a\text{R} + (n + 1)\,\Delta}{2a\text{R} + (n - 1)\,\Delta} \times \text{R} \text{ .... } [7]$$

But, if $\Delta = az + bz^2$ be treated as a quadratic, and we make in the result $\sqrt{a^2 + 4b\Delta} = q$, we have

$$z = \frac{q - a}{2b} \ldots\ldots\ldots\ldots\ldots\ldots [8]$$

which is the favorite *irrational method* of HALLEY. To improve its efficacy, call the value thus found r, and, taking in two additional terms of equation 2, we shall have

$$z = r - \frac{(c + dr)\,r^3}{q} \ldots\ldots\ldots\ldots\ldots [9]$$

This amendment, which usually quintuples the original number of true places, is due to the same eminent mathematician, who also remarks, that, by successively substituting in these formulae the corrected values of Δ and r, the advantage may be carried to any extent, without finding new values for a, b, c, &c.

Tho' it is perhaps a digression from the general train of this investigation, yet it will make the syllabus of theorems more complete if we observe here, that when ϕx is simply $x^n = N$, which produces $\Delta = N - R^n$, $a = nR^{n-1}$, $b = \frac{1}{2} n . \overline{n-1} . R^{n-2}$ &c, we derive the following formula:

From eq. 5 or 6, $$z = \frac{2\,(N - R^n)\,R}{(n+1)\,R^n + (n-1)\,N} \ldots\ldots [10]$$

equivalent to Halley's rational formulae for approximating towards the root of a pure power.

From eq. 7, or by adding R to the preceding expression, and reducing,

$$x\,(= R + z) = \frac{\overline{n+1} . N + \overline{n-1} . R^n}{\overline{n-1} . N + \overline{n+1} . R^n} \ldots\ldots\ldots\ldots\ldots [11]$$

the well-known theorem of HUTTON, for the same purpose.

From eq. 8, after the proper reductions,

$$z = \frac{1}{n-1}\sqrt{\frac{nN + (n-1) . (N - R^n)}{nR^{n-2}}} - \frac{R}{n-1} \ldots [12]$$

or, adding R,

$$x = \frac{\overline{n-2} . R}{n-1} + \frac{1}{n-1}\sqrt{\frac{nN + (n-2) . (N - R^n)}{nR^{n-2}}} \quad [13]$$

equivalent to Halley's irrational formulae, but more commodiously expressed. Each of these theorems admits of various enunciations, by combining any of the symbols. N, R, Δ, $\bar{z}\ (= \mathrm{N} + \mathrm{R}^n)$.

But, to return and conclude, the accurate value of z is

$$z = \frac{z\Delta}{\phi(\mathrm{R} + z) - \phi\mathrm{R}};$$

and, if the approximate value r be here introduced in the place of z, it gives

$$z = \frac{r\Delta}{\phi(\mathrm{R} + r) - \phi\mathrm{R}} \quad \ldots\ldots\ldots\ \ldots[14]$$

which exhibits the method of *Double Position.*

Concerning these epitomes, of all the approximating rules which are adopted for general use, the following observations naturally occur:

1. Whenever Newton's method fails, in consequence of an ambiguous or inadequate assumption for R, Halley's rational method must also fail; since it is only a corollary to the former. The same may be said of more complicated theorems, such as those in Simpson's Tracts and Algebra, which proceed on the supposition that $\frac{\Delta}{a}$ is a distinct approximation to the value of z.

2. The coefficients arising from substituting $\mathrm{R} + z$ for the unknown quantity x in any proposed equation, are respectively equal to the quantities arising from substituting R for x, either in Newton's limiting equations, or in the differential coefficients (so called) which are derived from the given expression in x by taking the successive fluxions, making at each step $\dot{x} = 1$, and dividing by the exponent of the step. For equation 3 shews these three views to be identical.

In practice, the first mode of obtaining these coefficients will be naturally adopted in finite equations, when R, a good conjectural value of x, is known; the second may be necessary, when R is ambiguous; the third, in case of surds and transcendental expressions.

3. Hence it follows that, excepting of course, the rules for the roots of pure powers, none of the methods above investigated are restricted to equations of any particular class; but their operation applies to transcendental and irrational quantities, as well as to such as are finite.

It does not however appear, that any one of the inventors was aware of the universal influence of his principles; and, tho' Dr. Brooke Taylor, a century ago, pointed out and exemplified this fact in regard of Halley's theorems, yet it would not be difficult to shew that, even at this day, the full value of those and the rest of our formulae is far from being generally adverted to.

A very probable cause of this neglect seems to be the lateness of the period at which fluxions are taught in our great seminaries. There is no apparent reason why the direct method should not be introduced (for example) between the 1st and 2nd parts of the Cambridge Elements of Algebra, so as to precede and assist the general theory of equations. But whatever may be thought of this suggestion, which is submitted with perfect deference, it is certain, that not only the convenience of the calculator, but in some measure the honor of English mathematics, is concerned in rendering these processes as familiar, as general, and as certain as possible.

This object has not been effected by the foreign mathematicians. LAGRANGE, in particular, whose avowed design was to substitute a perfect theory in place of the Newtonian methods, of which he has not failed to place the defects in the most prejudicial light, has, indeed, so far succeeded, that his theory is confessedly perfect; but that perfection consists altogether of a *beau idéal.* The nucleus, within which its practical applicability has scope, cannot be said to extend beyond cubic equations; and, even placing out of consideration his tedious investigation of limits, what is the rest of his process but a solution of numerical equations by means purely algebraical, and without the aid of any of the facilities peculiar to numerical notations? The rival methods present a contrast, similar to that between an active intelligent man, and a beautiful statute, standing secluded in the garden, and immoveable on its pedestal.

The real defect of Newton's method, and the rest of the same class, does not consist, as Lagrange intimates, in our uncertainty whether the assumptive value is a distinct approximation or not. Newton's solution, like that of his critic, comprizes two distinct operations; and, in case of ambiguity, it is perfectly invidious to contemplate his method of approximation separately from his method of limits. The commencement of every operation of reason, is of a class totally distinct from its ulterior prosecution, and depends either on observation or on experiment. This is true of the most elementary processes of arithmetic; and the utmost which it is not chimerical to attempt in regard of difficult equations, is to render the preparatory experiments as simple as we can. In this respect, the English author has a decided superiority.

In what then consists the defect? Clearly, in this; that, professing numerical solution, it nevertheless leaves us the necessity of recurring at stated intervals to the symbolic data, and, instead of one equation, obliges us to solve a succession of them. The French writers instead of removing this objection, have greatly increased it; so much so, that the number of digits in the decimal notation of the root of an equation may, in general, be accounted as equal to the number of transformed equations necessary for the solution, exclusively of the calculation of limits. In fact, the number of decimal places obtainable at n operations, by the methods of Lagrange and of Newton, are nearly as $n : 2^n$; not to mention that the very labor of reducing Lagrange's continued fraction to a decimal form is nearly equivalent of itself to that of Newton's whole operation.

A numerical solution ought, at least after the first transformation, to proceed in an uninterrupted tenor, without once recurring to algebraic notation. This is what BUDAN professes to have accomplished; but his method, more tedious, if possible, than that of his distinguished master, is properly nothing but an improved mode of getting the results of Newton's limiting equations. In fact, it may advantageously be substituted for the Newtonian method of limits, when it is necessary to scrutinize them; but, as an ulterior approximation, it is quite nugatory. Not only every digit of the root, but every unit of every digit, costs a new transformation.

The arithmetical practice of division and square root, are instances of a continuous mode of solving equations, $az = \triangle$ and $z^2 + 2Rz = \triangle$, of the preceding investigation. If on arriving at the higher powers, calculators have been at fault, and adopted recurring approximations, it has been because the general theory of evolution was not properly unravelled. This theory and that of equations in general, are intimately connected, or rather are one and the same.

These ideas will be found fully developed, in a paper composed by the writer of these strictures, and honored by insertion in the Philosophical Transactions, for this year; to which he humbly solicits the attention of mathematicians. By combining the improvements of preceding analysts, he has also endevored to simplify the questions of limits and possibility. In a word, he has labored to render the solution of an equation, from first to last, a purely arithmetical process, direct and certain, easy to be remembered, and easy to be practiced.

Bath, 1819.

End of the second part of the fourth Volume.

CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the Degree of B. A. in January, 1814.

BY THE TWO MODERATORS.

MONDAY, JANUARY 17, 1814.

MONDAY MORNING.—Mr. BLAND.

First and Second Classes.

1. The time of a body's falling through half radius by the uniform action of the centripetal force in the circumference of a circle is to the periodic time as radius is to the circumference of the circle. Required a proof.

2. Prove, without resolving the equation into factors, that if two numbers, a and b, when substituted for the unknown quantity in the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, give results affected with contrary signs, there is at least one root between a and b.

3. If a line intercepted between the extremity of the base of an isosceles triangle, and the opposite side (produced if necessary) be equal to a side of the triangle, the angle formed by this line and the base produced is equal to three times either of the equal angles of the triangle.

4. Given the greatest range of a projectile upon an horizontal plane; determine at what distance from the point of projection an object whose perpendicular height is (d), must be situated, so that the projectile may just strike the top of it.

5. From the vertex of a paraboloid of given dimensions, a part equal to one-fourth of the whole is cut off by a plane parallel to the base; and the frustum being then placed in a fluid with its smaller end downwards, sinks till the surface of the fluid bisects the axis which is vertical. It is required to determine the specific gravity of the paraboloid, that of the fluid in which it is immersed, and the density of the atmosphere being given.

6. Given the fluent of $\frac{d\dot{z}}{z \cdot (a + cz^2)^2} = A$, to find the fluent of $\frac{d\dot{z}}{z^3 \cdot (a + cz^2)^2}$, and find the algebraical relation of x to y, in the equation $y^2\dot{y} = 3yx\dot{x} - x^2\dot{y}$.

7. Find the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} +$ &c. in inf. and $\frac{10 \cdot 18}{2 \cdot 4 \cdot 9 \cdot 12} + \frac{12 \cdot 21}{4 \cdot 6 \cdot 12 \cdot 15} + \frac{14 \cdot 24}{6 \cdot 8 \cdot 15 \cdot 18} +$ &c. to n terms.

8. Required the nature of the curve such, that if any point P in it be taken, and an ordinate PN and normal PG be drawn to the axis; then if the triangle PNG be placed in such a position that the subnormal NG, may become the ordinate, PG will be the normal.

9. To spectators situated within the tropics, the Sun's azimuth will admit of a maximum twice every day, from the time of his leaving the solstice till his declination becomes equal to the latitude of the place. Required a proof.

10. If parallel rays fall upon a single thin lens of given substance; determine the diameter of the least circle into which all the rays of different colours are collected, the linear aperture of the lens being known.

11. Compare the magnitude of that part of the disturbing force of the sun on the moon, which acts perpendicular to the plane of the moon's orbit, with the moon's gravity to the earth.

12. The velocities of two bodies A and B are in a given ratio, and they begin to move at the same time from A and B, the extremities of a given line AB; A moving uniformly in a direction inclined at a given angle to AB, and B uniformly in the direction BA. Determine the nature of the curve to which the line joining the bodies is always a tangent.

13. The moon's nodes complete a revolution in about 19 years. Determine the periodic time of the nodes of the third satellite of Jupiter, which revolves in about seven days, Jupiter's period being about 12 years.

14. If (a) be the number of chances for the happening of an event, and (b) the number for its failure in each single trial; find the probability of its happening p times and failing q times in $(p + q)$ trials; and determine how many trials are necessary to make it an even chance whether the event will happen or not.

MONDAY AFTERNOON.—MR. BLAND.

Fifth and Sixth Classes.

1. Extract the square root of $\frac{9\cdot0240160}{25\cdot3009}$.

2. Prove the rule for completing the square in a quadratic equation.

3. Find the sum of the series

$$-9-7-4-\&c. \text{ to 20 terms.}$$

$$\text{and } 1+\frac{2}{3}+\frac{4}{9}+\&c. \text{ to 10 terms.}$$

$$\text{and } 1\cdot2+2\cdot3+3\cdot4+\&c. \text{ to } n \text{ terms.}$$

4. If three quantities are in an increasing arithmetical progression; shew that the second will have to the first a greater ratio than the third to the second.

5. The weights of two perfectly elastic balls are 11 and 8, and their velocities in opposite directions are 12 and 7. Required their velocities after impact.

6. Let the height of an inclined plane be (*a*) feet, and its length (*c*) feet. Find the time of a body's descending down (*a*) feet of the plane.

7. Two fluids, whose magnitudes and specific gravities are given, being mixed together; the magnitude of the mixture : sum of the magnitudes of the ingredients :: *n* : 1. Determine the specific gravity of the mixture.

8. Compare the time in which any prismatic vessel is emptied by an orifice in the lower surface, with the time of a heavy body's falling through a space equal to twice the depth of the orifice.

9. Divide a right line into two parts such, that their rectangle may be equal to a given square; and determine the greatest square that the rectangle can equal.

10. Find the fluxions of $(a+cz^n)^m \times z^p$, and of $x \times \frac{\sqrt{(1+x^2)}}{\sqrt{(1-x^2)}}$, and find the fluents of $\frac{\dot{x}}{x\sqrt{(x^2-a^2)}}$, $\frac{x^3\dot{x}}{a^2-x^2}$, and $x^2\dot{x}\sqrt{(a^2+x^2)}$.

11. Construct Newtons's Telescope and find its magnifying power.

12. Explain the reason why the order of the colours is inverted in the secondary rainbow.

13. Given the sun's altitude at six o'clock, and also when east; find the latitude of the place.

14. If from a quantity which varies as $\frac{1}{A^2}$, any quantity be subtracted which varies as A, the remainder will vary in a higher inverse ratio than the inverse square of A; but if to a quantity varying as $\frac{1}{A^2}$ another be added which varies as A the sum will vary in a lower inverse ratio than the inverse square of A. Required proof.

15. Find the law of the force tending to the centre of the logarithmic spiral.

16. Prove that when the force acts in parallel lines, the velocity in the direction perpendicular to the direction in which the force acts, is constant.

17. If the altitude of a cylinder be equal to the diameter of its base, the whole surface is six times the area of the base.

18. If $a^{mx} b^{nz}$ be constant, and $(mx + n) \times (nz + m)$ be a maximum; prove that $a^{mx+n} = b^{nz+m}$.

MONDAY AFTERNOON.—Mr. MACFARLAN.

Third and Fourth Classes.

1. Required the perpendicular from the vertex upon the base of a triangular pyramid, all the sides of which are equilateral triangles of a given area.

2. Given the difference of the times of setting of two stars whose declinations are known; find the latitude of the place.

3. Find the centre of oscillation of a conical surface suspended by the vertex; and find the ratio between the radius of the base and the axis, when the centre of oscillation is in the base.

4. The length of a pendulum oscillating seconds on the earth's surface being given; find the length of a pendulum oscillating seconds at the distance of the earth's radius from the surface. Then determine a point below the surface where the last pendulum will vibrate in the same time.

5. Two roots of the equation $x^4 + x^3 - 11x^2 + 9x + 18 = 0$ are of the form $+ a, - a$. Find all the roots.

6. When the force varies as that power of the distance whose index is $(n - 1)$. Shew that the velocity of a body falling from rest varies as $\sqrt{\frac{(P^n - A^n)}{n}}$ where P is the greatest and A

the variable distance. And find the value of this expression when the force varies inversely as the distance.

7. If from the extremity of the diameter of a circle tangents be drawn and produced to intersect a tangent to any point in the circumference, the right lines joining the points of intersection and the centre of the circle shall form a right angle.

8. Sum the series $\frac{1}{1\cdot3} - \frac{1}{2\cdot4} + \frac{1}{3\cdot5} - \frac{1}{4\cdot6} +$ &c. to n terms, and $\frac{1}{2\cdot3} + \frac{1}{4\cdot5} + \frac{1}{6\cdot7} + \frac{1}{8\cdot9}$ ad inf.

9. Find the fluents of $\frac{\dot{x}}{\sin z \times \cos z}$ and $\frac{\dot{x}}{\sqrt{(A + Bx - Cx^2)}}$.

10. Find the attraction of a sphere on a particle of matter placed at any distance from the centre, the force of each particle varying inversely as the cube of the distance.

11. Find the equation to the curve, the length of whose tangent between any point and the axis is a constant quantity.

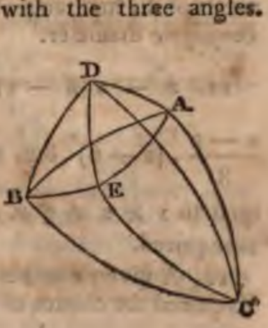
12. The equation to a curve is $y^3 - axy + x^3 = 0$. Find the value of the ordinate when a maximum, and the corresponding value of the abscissa; and shew that the above is a maximum and not a minimum.

13. A paraboloid placed with its vertex downwards being full of water, is supplied at a given rate. There is a small hole in the vertex, which, when the vessel is full, would discharge n times the quantity supplied. Required the altitude at which the surface remains stationary, and the time elapsed before this takes place.

MONDAY EVENING.—Mr. MACFARLAN.

1. A body placed in the centre of gravity of a triangle is acted upon by three forces represented in quantity and direction by the lines joining the centre of gravity with the three angles. Shew that the body will remain at rest.

2. The sides of the spherical triangle ABC are each a quadrant. D and E (any two points on the surface of the sphere) are joined by the arc of a great circle. Shew that the cosine of DE is equal to the cos AD × cos AE + cos BD × cos BE + cos CD × cos CE.



3. If the sum of the odd digits in a number be $11m + e$ and of the even $11n + e$, this number being divided

mon pack are dealt in order, one by one, stake each 1*l*. with the condition, that he to whom the first knave is dealt, shall be the winner. What is the value of A's expectation?

21 Find the curve by the revolution of which round an axis the solid will be formed, which shall attract a particle placed at its vertex with the greatest possible force, the force of each particle varying inversely as the square of the distance.

22. A cylindrical vessel full of water is balanced by a weight P over a fixed pulley. A hole of given dimensions being made in the bottom, it is required to find how far P will descend during the time of emptying.

23. Prove that the sum of the reciprocals of the prime numbers is an infinitely great number though infinitely less than the sum of the reciprocals of the natural numbers.

TUESDAY MORNING.—Mr. MACFARLAN.

First and Second Classes.

1. A person borrowed P*l*. at interest. To discharge this he invested 2*l*. at the end of the first year, 4*l*. at the end of the second, and 8*l*. at the end of the third, and so on. How many years will elapse before this fund be large enough to discharge the debt,—compound interest being allowed on both sides at a given rate?

2. Required the length of a spherical shell of iron, which, when filled with a fluid, shall just float in water; the specific gravities of iron, of water, and of the fluid being given.

3. Compare the length of a degree of latitude at any place on the earth's surface, with the length of a degree of longitude at the equator.

4. The inclination of a small tube in the side of a vessel of water being given, and its height above the horizontal plane; it is required, from observing the point of the plane struck by the stream, to assign the altitude of the water within the vessel; and to describe the whole track of the issuing fluid.

5. If round any point within the circumference of a circle (not being the center) equal adjoining angles be described; of the circumferences on which they stand, that which is nearer the diameter passing through the point is less than the more remote.

6. In a combination of two wheels and axles, the circumference of each wheel is n inches; of each axle 1. A weight, P, is applied to the circumference of one of the wheels as a power to raise matter to a certain height. How much must be raised

each time, that the whole quantity may be raised in the least time possible?

7. Of all cones containing a given quantity of matter, to find that which attracts a particle placed at its vertex with the greatest possible force.

8. Show that when a quantity is a maximum or a minimum, the first fluxion vanishes; and that the quantity is a maximum or a minimum, accordingly as the second fluxion is negative or positive.

9. An imperfectly elastic ball is projected with a given velocity against an hard horizontal plane, and being reflected, just reaches the point of projection in t''. Required the distance of the plane, from the point of projection, and the elasticity of the body.

10. A cylindrical tube of given length, closed at one end, being let down in a vertical position into the sea, it was observed what part of the tube the water occupied. It is hence required to assign the depth, 33 feet of sea water being assumed to measure the weight of the atmosphere. How must this tube be graduated to be used as a gauge to measure depths in the sea?

11. Find the length of a common parabola, and deduce Cotes's construction.

12. If $x = \frac{\varepsilon^{nz} - \varepsilon}{2a}$, where a and b are roots of the equation, $v^2 + 1 = 0$.

$$\text{Shew that } \dot{z} = \frac{\dot{x}}{\sqrt{(1 - x^2)}}.$$

13. Sum the series $1^2 + 3^2 + 7^2 + 15^2$ &c. to n terms, and

$$\frac{1}{2 \cdot 3} \times \frac{1}{2} + \frac{1}{3 \cdot 4} \times \frac{1}{2^2} + \frac{1}{4 \cdot 5} \times \frac{1}{2^3} + \text{\&c. ad inf.}$$

TUESDAY AFTERNOON.—Mr. MACFARLAN.

Fifth and Sixth Classes.

1. Find the value of £·12341414141 &c.

2. The amount of £500 in $\frac{1}{4}$ of a year was £520. Required the rate per cent.

3. Find the circumference of a circle whose radius is unity.

4. Sum the following series,

(1). $2 + 3 + \frac{9}{2} +$ &c. to 20 terms, (2). $\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+$ &c. to n terms,

and $\frac{1}{1\cdot2\cdot3}+\frac{1}{3\cdot4\cdot5}+$ &c. ad inf. by increments.

5. Solve the following equation whose roots are in arithmetic progression; $x^3-9x^2+23x-16=0$.

6. Find the fluents of $\frac{a\dot{x}}{a^2-x^2}$, $\frac{x^2\dot{x}}{\sqrt{(a^2+x^2)}}$, and the fluxion of $\frac{(x+a)^2}{\sqrt{(x^2-a^2)}}$.

7. The arc of a circle which a body, acted upon by a centripetal force, uniformly describes in any given time is a mean proportional between the diameter of the circle, and the space described by a heavy body from rest in the same time when urged by the force in the circumference continued uniform.

8. Show that the logarithm of $(1+u)$ is equal to $u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\frac{u^5}{5}-$ &c.

9. Given the radii of the surfaces of a double convex lens and the ratio of the sines of incidence and refraction. Find the focal length.

10. Find the latitude of the place at which the sun sets at three o'clock on the shortest day.

11. When the force varies as the distance, the periodic time in all ellipses round the same centre are equal.

12. A body (A) weighs 12lbs. in vacuo, and 9lbs. in water; another body (B) weighs 10lbs. in vacuo, and 8lbs. in water; compare their specific gravities.

13. If the number of mean proportionals interposed between two elastic bodies A and X be increased without limit, the velocity of A will be to the velocity communicated to X by means of the intermediate bodies :: $\sqrt{X} : \sqrt{A}$.

14. The apparent diameter and declination of the sun being given, find the time of his transit over the meridian.

15. The plane of a circle being vertical, and any number of chords being drawn to the lower extremity of the vertical diameter; Find the locus of any number of heavy bodies falling together from the upper extremities of the diameter and the chords at any given instant of time.

16. If any number of projectiles be thrown at the same instant from the same point and with equal velocities, but in seve-

ral directions in the same vertical plane, they will at the expiration of any time all be found in the circumference of some circle.

TUESDAY AFTERNOON.—Mr. BLAND.

Third and Fourth Classes.

1. Shew from the principles of the fifth book of Euclid, that a ratio of greater inequality is diminished, and of less inequality increased by adding a quantity to both its terms.

2. The time of day at a given place determined from observations of the sun's altitude is $9^h\ 10^m\ 45^s$; and a chronometer set to Greenwich time shews $6^h\ 3^m\ 10^s$. Required the longitude of the place of observation from Greenwich.

3. In any harmonic progression, the product of the two first terms is to the product of any two adjacent terms as the difference between the two first is to the difference between the two others. Required a proof.

4. An object is placed between two plane reflectors which are inclined to each other at an angle of 60°. Determine the whole number of images formed by the reflectors.

5. If the greatest possible rectangle be inscribed in the quadrant of a given ellipse, shew that the elliptic areas cut off by the sides of the rectangle are equal.

6. Prove that an equation of an odd number of dimensions, and an equation of an even number if its first and last terms be of different signs, must have at least one real root.

7. Find the sums of the series

$$1 . 2 . 4 + 2 . 3 . 5 + 3 . 4 . 6 + \text{\&c. to } n \text{ terms},$$

$$\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \text{\&c. to } n \text{ terms}.$$

and $1 . 2 + 2 . 3 . x + 3 . 4 . x^2 + \text{\&c. in inf.}$

8. If the force vary inversely as the square of the distance, and a body be projected at a given angle with a velocity which is to the velocity in a circle at the same distance, as $\sqrt{2} : 1$. Determine the nature of the orbit described.

9. Prove that the surface of any segment of a sphere cut off by two parallel planes is to the whole surface of the sphere as the intercepted portion of the diameter is to the whole diameter.

10. Find the fluent of $\dfrac{(x-1) . \dot{x}}{(x^2+1)^2}$; construct the fluent of

$\frac{x\dot{x}}{x^2 - 2ax + 1}$, where *a* is less than unity; and shew that the fluent of $\frac{d\dot{z}}{\sqrt{(a^2 - bz^2)}} = \frac{d}{\sqrt{b}}$ × circular arc whose sine is $\frac{z\sqrt{b}}{a}$, and radius $= 1$.

11. A paraboloid whose vertex is downwards is filled with water to a given altitude. Having given the diameter of the upper surface, find what ought to be the diameter of the hole at the bottom, so that the upper surface may descend through a given space in a given time.

12. If the force varies as the distance, and two bodies fall towards two different centres of force; compare the velocities at any point of their descent.

13. Two elastic balls beginning to descend from different points in the same vertical line, impinge on a perfectly hard plane inclined at an angle of 45°, and move along a horizontal plane with the velocities acquired. Shew that if a circle be described, passing through the two points from which the balls began their motion, and touching the horizontal plane, the point of contact will bisect the distance between the vertical line and the point where they impinge on each other.

14. Given, that the distance of the centre of gravity of an area from its vertex is an *n*th part of the abscissa; find the distance of the centre of gravity of the solid formed by the revolution of this area round its axis.

15. Determine the proportion between the radius of the globe and wheel, when the length of the cycloid within the globe, (Sect. 10) is a maximum.

16. If centripetal forces tend to the several points of spheres, proportional to the distances of those points from the attracted bodies, the compounded force with which two spheres will attract each other mutually is as the distance between the centres of the spheres.

TUESDAY EVENING.—MR. BLAND.

1. The sum of *n* arithmetic means between 1 and 19 is to the sum of the first $n - 2$ of them :: 5 : 3. Find the means.

2. Two equal weights are connected by a string passing over a fixed pulley. Supposing a weight to be added on one side, and the length and weight of the string, and the difference of the altitudes of the weights at the commencement of the motion to be given; determine in what part of the descent, the velocity will be neither increased nor diminished by the string's weight.

3. If the abscissa of a curve bear a finite ratio to the ordinate, prove that the abscissa will cut the curve in a finite angle.

4. The place of the node and the inclination of the moon's orbit to the plane of the ecliptic being given; find the place of the moon when her declination is the greatest possible.

5. Find the value of $\frac{\sqrt{(2a^3x - x^4)} - \sqrt{(ax^3)}}{a - \sqrt{(ax)}}$, when $x = a$; and find the fluxion of the hyp. log. $\frac{\sqrt{(a + x)} + \sqrt{(a - x)}}{\sqrt{(a + x)} - \sqrt{(a - x)}}$.

6. If, in a circle a straight line be drawn cutting the diameter at any angle (A); prove that the difference of the segments of the diameter will be to the difference of the segments of the line as the diameter is to the chord of an arc, which measures twice the complement of A.

7. If, from the extremity of the major axis of an ellipse which is perpendicular to the horizon, chords be drawn making with it angles of 75° and 45°, and from the points where the chords meet the curve, ordinates be drawn to the axis; the square of the time down the first chord will be to twice the square of the time down the second in the subduplicate ratio of the rectangles under the segments of the axis made by the ordinates.

8. If a square be inscribed in circle and another circumscribed about both: compare the pressures upon the circle and the squares when immersed vertically in a fluid, the angular point of the circumscribing square coinciding with the surface of the fluid.

9. A hollow cone whose vertical angle is 60°, is filled with water, and placed with its base downwards. It is required to determine the place where a small orifice must be made in its side, so that the issuing fluid may strike the horizontal plain in a point, the distance of which from the bottom of the vessel is to the distance of the orifice from the top as 5 : 4.

10. The distance of the centre of gravity of the surface of a solid from the vertex is equal to half the abscissa; determine the nature of the curve by the revolution of which round its axis the surface was generated.

11. If two equal parabolas be placed in such a manner that they may touch each other at the vertices, and one be made to roll upon the other, its focus will describe a right line, and the vertex a cissoid, the diameter of whose generating circle is equal to half the latus rectum of the parabola.

12. If a body revolve in an orbit round a centre of force, and at the same time the orbit revolve round the same centre in such a manner that the angular velocity of the body in the orbit if fixed, may be to its angular velocity when revolving, in the ratio

of F : G. Find the centripetal force necessary to retain the body in a revolving orbit, the force in the fixed orbit varying as the nth power of the distance. And apply this to the cases of an ellipse when the centre of force is in the focus, and when in the centre.

13. Let α, β, γ, &c. be the roots of the equation $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$, (m) of which are possible; shew that if the equation be transformed into one whose roots are $(\alpha-\beta)^2$, $(\alpha-\gamma)^2$, $(\beta-\gamma)^2$, &c. the last term of the transformed equation will be positive or negative according as $m \cdot \frac{m-1}{2}$ is an even or an odd number.

14. Find the fluents of $\frac{e_z z\dot{z}}{(1+z)^2}$; where e = the base of hyp. logs.; $\frac{dy^2\dot{y}}{(ar^2+by^2)\cdot\sqrt{(r^2-y^2)}}$, $\frac{\dot{x}}{\sin.^2 x \times \cos.^4 x}$, and $\frac{b\dot{y}}{\left\{(a^2-y^2)\cdot(a+y)\right\}^{\frac{2}{3}}}$.

15. Find the relation of x to y in the equations,

$$y^n\dot{y} - a^{n-1}x\dot{y} + c^{n-1}y\dot{x} = 0,$$

$$a^2\dot{y}^2 + bx\dot{y}^2 = c^2\ddot{y},$$

and determine the nature of the curve, in which $\frac{\dot{x}}{x} - \frac{\dot{y}}{y} : \frac{\dot{x}}{y} - \frac{\dot{y}}{x} :: n : 1$, x and y being the abscissa and ordinate.

16. Find the sum of the series,

$\frac{5}{1\cdot2\cdot3} + \frac{7}{2\cdot3\cdot4} + \frac{9}{3\cdot4\cdot5} + \frac{11}{4\cdot5\cdot6} +$ &c. to n terms by increments,

$\frac{2}{3\cdot5} - \frac{3}{5\cdot7} + \frac{4}{7\cdot9} - \frac{5}{9\cdot11} +$ &c. in inf.

$\frac{\text{tang. A}}{1} - \frac{\text{tang. 2A}}{2} + \frac{\text{tang. 3A}}{3} -$ &c. in inf. and having given the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} +$ &c. in inf. find the sum of $\frac{1}{1^2\cdot2\cdot3} + \frac{1}{2^2\cdot3\cdot4} + \frac{1}{3^2\cdot4\cdot5} +$ &c. in inf.

17. The reflecting curve of a semicircle, and the radiating point is in the circumference; determine the nature of the caustic, its length, and the density at different points.

18. The latitudes of two places on the earth's surface are complements to each other, and on a given day the sun rises (n) hours sooner at one place than at the other; determine the latitude of each place.

19. If a system of bodies be connected together and supported at any point which is not the centre of gravity, and then left to descend by that part of their weight which is not supported; $4l$ multiplied into the sum of all the products of each body into the space it has perpendicularly descended will be equal to the sum of all the products of each body into the square of its velocity; l being $= 16\frac{1}{12}$ feet.

20. A ring of given weight descends by its gravity down the arc of *any algebraic curve*; and the curve revolves uniformly about its axis which is perpendicular to the horizon in t'': Determine the velocity of the ring at any point of its descent.

21. If the force of gravity be uniform, and act perpendicularly to the horizon, determine the path of a projectile in a medium, the resistance of which is proportional to the velocity of the body.

22. Having given the relation between the centrifugal force and the force of gravity at the earth's equator; determine the relation between the centrifugal force and the force of gravity at the equator of jupiter; the densities and times of revolution round their axes being known.

23. Shew that the mean motion of the nodes of the lunar orbit is not affected by the excentricity of the orbit; and that the true motion of the nodes in an elliptic orbit is equal to the motion of the nodes in a circular orbit when the radius vector is a mean proportional between the semi-axes major and minor.

24. If a hyperboloid and a cone be generated by the revolution of an hyperbola and its asymptotes; and the cone being excavated and placed with its axis vertical, water be poured into it till the surface touches the vertex of the hyperboloid; shew that whatever be the inclination of the axis, the surface of the water will always be a tangent plane to the hyperboloid.

CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the degree of B. A. in January, 1815.

BY THE TWO MODERATORS.

MONDAY, JANUARY 16, 1815.

MONDAY MORNING.—Mr. HUSTLER.

First and Second Classes.

1. If A be any arc whatever, prove that cosec. A + cosec. 2A + cosec. 4A + &c. to n terms $= \text{cot.}\ \frac{A}{2} - \text{cot.}\ 2^{n-1} A$.

2. Shew that the sun's rising is least accelerated by refraction at the time of the Eqinoxes,

3. If an hyperbola and its asymptotes revolve about the axis major, prove that all the sections of the cone made by planes touching the hyperboloid have the same axis minor, which is the axis minor of the hyperbola.

4. Near the solstice the variations of the sun's declination are as the squares of the variations in longitude nearly.

5. Find from Taylor's Theorem the arc in terms of its *cosine.*

6. Having given the refracting powers of two mediums, find the ratio of the focal lengths of a convex and concave lens formed of these substances, which, when united, produce images nearly free from colour.

7. If $t =$ tangent of half the angle ASP (Newton, Sect. 6.) to $r = 1$, shew that the parabolic area ASP $= a^2 \times \left\{\frac{t^3}{3} + t\right\}$, where A is the focal distance of the vertex.

8. If a caustic be formed by a reflecting curve, shew that the reflected ray is always a tangent to the caustic.

9. In any recurring series $a + bx + cx^2 +$ &c. whose scale of relation is $f + g + h +$ &c. if any row of the differences of $a, b, c,$ &c. $= 0$, prove that $f + g + h +$ &c. $= 1$.

10. If a cylinder be cut by two parallel planes intercepting a given part of the axis, shew that the solidity between the planes is the same whatever be the inclination of the planes to the axis.

11. Find the fluents of

$$\frac{\dot{x}}{x^3\,(a^2 - x^2)} \text{ and of } x^3\dot{x}\sqrt[3]{(a^2 + x^2)}.$$

12. Having given the latitude of the place and the moon's declination, determine the height of the superior and inferior tide, and compare the height of the tide at the equator and pole of the earth, when the moon's declination is 30°.

13. If the moon's orbit be considered circular, and the position of the nodes be given, show that, when the *horary* motion of the node is a maximum, the moon's distance from the quadratures equals half the node's distance.

14. A semicubical parabola moves in its own plane, with its axis always coinciding with the same line. Determine the nature of a curve which, beginning at the vertex of the parabola in its first position, is perpendicular to it in all positions.

MONDAY AFTERNOON.—MR. HUSTLER.

Fifth and Sixth Classes.

1. If four quantities be in geometric progression, the sum of the two extremes is greater than the sum of the two means.

2. From the equation $a^{mx} = b - a^{mx-1}$ find the value of x.

3. If the interior angle BAC and the exterior angle DAC of any triangle ABC be bisected by lines AE, AF which also cut BC in E, F, show that BF, BC, and BE are in harmonic progression.

4. Required the number of guineas with which four persons A, B, C, D respectively begin to play, if after A has won half of B's, B a third of C's, and C a fourth of D's, each has twelve guineas.

5. What power acting parallel to the length of an inclined plane whose elevation is 30°, will draw a given weight Q, 40 feet up the inclined plane in 5 seconds?

6. The latus rectum of an ellipse, being produced both ways to meet the circle described on the axis major, = axis minor.

7. Given the cot A and cot B, find cot (A ± B) radius being = 1; and adapt the expression to radius r.

8. Form the biquadratic equation, two of whose roots are $1 + \sqrt{(a^3)}$ and $-\sqrt{(-b)}$.

9. How far must a body fall internally towards the focus of an hyperbola to acquire the velocity in the curve?

10. The sun's altitude at any time being 30°, find the position of a stick of given length, so that the shadow may be the longest possible; and find the length of the shadow at that time.

11. Find the solid traced out by a curve whose equation is $y^2 = \frac{b^2 x}{a - x}$.

12. If a billiard table be in the form of an ellipse, and a perfectly elastic ball be struck from either focus in any direction, it will return after two reflections from the curve to the same point.

13. An object at the bottom of a vessel appears to change its place when water is poured into the vessel. Explain this circumstance.

14. Given the specific gravities of wood and water :: 2 : 3, to what depth would a given paraboloid of wood sink when immersed with its vertex downwards?

15. Find the fluxions of $\frac{a + x}{a^2 + x^2}$ of $x . (a^2 + x^2) . \sqrt{(a^2 - x^2)}$ and of the secant of x;

also the fluents of $\frac{\dot{x}\sqrt{(a^2 - x^2)}}{x}$ and of $\frac{b\dot{x}}{\sqrt{(1 - ax^2)}}$.

16. At a given place, and on a given day, find the point of the horizon where the sun rises,

17. Shew, according to Newton's second section, that if a parabola be described by a force tending parallel to the axis to a point indefinitely distant, the force must be constant.

MONDAY AFTERNOON.—MR. BLAND.

Third and Fourth Classes.

1. Extract the square root of

$$ab - d^2 + 4c^2 \pm 2\sqrt{(4abc^2 - abd^2)}.$$

2. The sum of an even number of terms of any arithmetic progression, whose common difference is equal to the least term, will be four times the sum of half that number of terms diminished by half the last term.

3. The greatest and least corrected zenith distances of a circumpolar star being 38° 19′ 43″, and 34° 53′ 49″; determine the latitude of the place of observation.

4. Two forces acting upon a body in the same or in opposite directions, will cause it to move with a velocity equal to the sum or difference of the velocities which it would have received from the forces separately. Required a proof.

5. If bodies fall towards different centres of force, from different altitudes, compare the times of descending through any space; $F \propto \frac{1}{(\text{dist.})^2}$.

6. The length of the subtangent to the cissoid being equal to one-fourth of the diameter of the generating circle; determine the point in the curve from which the tangent is drawn.

7. Transform the equation $x^3 - px^2 + qx - r = 0$ into one whose roots shall be mean proportionals between the roots of the equation, and a given quantity (m).

8. One side of a cubical vessel of water of given dimensions being loose; find the position, magnitude, and direction, of a single force which shall keep it at rest.

9. Find the time in which a vessel formed by the revolution of a given logarithmic curve round its asypmtote, will empty itself through a given orifice in the bottom; the length of the axis and extreme ordinate being known.

10. If the force of gravity $\propto \frac{1}{(\text{dist.})^2}$ from the centre; find according to Newton's method, the absolute velocity in feet, and absolute time in seconds, of descending through any space towards the centre.

11. Find the fluents of $z\dot{z}\sqrt{\frac{a+z}{a-z}}$ and $\frac{\dot{z}}{z} \times (a^2+z^2)^{\frac{3}{2}}$; and shew that the fluent of $\frac{\dot{z}}{(1-az^2)\cdot\sqrt{(1-z^2)}} = \frac{1}{\sqrt{(1-a)}} \times$ circular arc, whose radius $= 1$, and cosine $= \sqrt{\frac{1-z^2}{1-az^2}}$.

12. The focal length of a double convex lens, whose thickness is inconsiderable, and whose surfaces have the same curvature, is equal to the diameter of one of the surfaces. Determine the ratio of the sines of incidence and refraction.

13. The areas of unequal ellipses are in a ratio compounded of the subduplicate ratio of their parameters, and the sesquiplicate ratio of the principal axes. Required a proof.

14. If a body be acted upon by two forces which vary according to different laws of its distance from the centre, as the pth and qth powers; determine the angle described, while it passes from one apse to the other, the orbit described being nearly circular, and the forces at the apse being as $1 : n$.

15. Let a plane isosceles triangle vibrate edgeways, suspended by its vertex. At what distance from its vertex must it strike an immoveable obstacle, so that its motion in the plane of vibration may be destroyed?

16. If two similar mediums are separated from each other by a space terminated on each side by parallel planes; and a body in its transit through this space, is attracted or impelled perpen-

dicularly towards either medium, and is not agitated or hindered by any other force; and the attraction is every where the same at equal distances from either plane, taken towards the same side of the plane; prove that the velocity of the body before incidence is to its velocity after emergence as the sine of emergence to the sine of incidence.

MONDAY EVENING.—MR. BLAND.

1. If (a) and (b) be the two first terms of a decreasing geometric progression, the sum of the series in inf. is $= \dfrac{a^2}{a - b}$. Required a proof.

2. If a tangent be drawn to any point of an ellipse, and from the point of contact a straight line be drawn to either focus; this shall be parallel to the straight line drawn from the centre to the intersection of the tangent and perpendicular from the other focus.

3. The moon's longitude at noon at Greenwich, Jan. 1815, is

on the 16th......	0°	16′	48″
17th......	12	46	55
18th......	25	35	31
19th......	38	46	4.

Find its longitude on the 17th, at 6 o'clock.

4. In a given latitude, determine the vertical circle in which the difference of the altitudes of the sun in any two given days shall be a maximum.

5. If a body revolve in a reciprocal spiral, the force tending to the centre; prove that the times of its moving through successive angles of 180°, are in the proportion of the numbers $\dfrac{1}{1 \cdot 2}$, $\dfrac{1}{2 \cdot 3}$, $\dfrac{1}{3 \cdot 4}$, &c.

6. Prove that every odd cube number is equal to the sum of as many terms of a series, which have a common difference unity, as its root contains units, the middle sum of the series being the square of the root.

7. Find the value of $\dfrac{1}{x + a} + \dfrac{a}{(x + a) \cdot (x + b)} + \dfrac{ab}{(x + a) \cdot (x + b) \cdot (x + c)}$ + &c. to n terms;

and of $\dfrac{4}{1 \cdot 3} - \dfrac{12}{5 \cdot 7} + \dfrac{20}{9 \cdot 11} - \dfrac{28}{13 \cdot 15}$ + &c. in inf.

8. On the side of a vessel filled with fluid, let any number of

circles be so situated that the pressures on them may be as the cubes of their diameters; determine the ratio of their distances from the surface of the fluid.

9. If water ascend and descend in the erect legs of a cylindrical canal, alternately; determine the nature and dimensions of the curve described by the centre of gravity of the water in the legs.

10. Two chains of the same uniform thickness and density are suspended from two given points, and attracted towards a centre of force, the law of the force being any power or root of the distance. Shew that the pressures on the points of suspension are proportional to the squares of the velocities which would be acquired by bodies falling towards the centre from the points of suspension, down spaces which are equal to the lengths of the chains,

11. Trace the curve whose equation is $x^2 + y^2 = \frac{b^2x^2}{2ax-x^2}$, and find its area when $b = a$.

12. A perfectly elastic ball A falls from the upper extremity of a given vertical line AB, and at the same time another perfectly elastic ball B is projected upwards from a horizontal hard plane at the bottom; they meet in some point C, and are reflected back. Determine the point C, so that they may ascend and descend from it continually: and find the velocity of B at that point.

13. Find the fluents of

$$e^{x} \cdot (\dot{P} + P\dot{x}), \text{ of } \frac{dz^{pn-1}\dot{z}}{(a+cz^n)^m \cdot (e+fz^n)^r}, \text{ and of}$$

$$\frac{x^{m-1}\dot{x}}{(1-x^m) \cdot \sqrt[2m]{(2x^m-1)}}.$$

14. If the middle points of any two edges of a triangular pyramid which do not meet, be joined; shew that the middle point of the connecting line is the centre of gravity of the pyramid.

15. If parallel rays be incident on a spherical surface of a plano-convex glass mirror, whose thickness is a semi-diameter and a half of the spherical surface, prove that they will, after having been refracted at the convex and reflected at the plane surface, converge to that point where the axis intersects the convex surface.

16. Two plane reflectors being inclined to each other at a given angle, determine the diameter of a circular arc, in which a luminous object must move between them, so that the ray, which has been reflected by any given point of one of them, may, after reflection at the second plane, always intersect the arc in the point in which the object is.

17. If the circle of curvature to the vertex of a parabola be described, and another circle touch that, and the arms of a parabola; and so on continually; prove that the radii of the circles will be in the proportion of the numbers, 1, 3, 5, 7, 9, &c.

18. If the force vary according to any law of the distance; shew that in any orbit, at the point where the centripetal and centrifugal forces are equal, the velocity towards the centre of force is a maximum.

19. Determine the nature of the curve by the revolution of which round its axis a solid will be generated, such that a corpuscle placed on its surface will be attracted towards the centre of gravity with a force varying as the distance; the solid revolving round its axis in a given time.

20. Find the horary increment of the area, which the Moon, by a radius drawn to the Earth, describes in a circular orbit. (Newton, Book III. Prop. 26.)

21. The excentricity of P's orbit (Sect. XI.) being small; find the variation of the major axis in a whole revolution of P, if the force at P be augmented or diminished by a small quantity in the ratio of $1 : 1 \pm n$.

22. If the co-latitude of the place of observation be equal to the Moon's declination, or less than it, there will be no inferior or no superior tide, according as the latitude and Moon's declination are of the same or different denominations.

23. Let a spherical body descend from rest in a fluid whose specific gravity is to that of the body as $1 : n$. Determine the velocity of the sphere at any point of its descent; and shew that the greatest velocity which it can acquire is equal to that which would be acquired by it in descending from rest, in vacuo, by the constant force of its comparative gravity through a space which is to $\frac{4}{3}$ of its diameter $:: n : 1$.

24. A given cylindrical rod falls by gravity towards a horizontal plane, whilst at the same time its extremity moves freely along the plane: determine the pressure upon the plane in any position; and the velocity of the moving point.

TUESDAY MORNING.—MR. BLAND.

First and Second Classes.

1. If the terms of the series arising from the expansion of $(a + b)^n$ be multiplied respectively by the terms of the arithmetic series $0x$, $1x$, $2x$, $3x$, &c. find the sum of the resulting series.

2. If a polygon has $(n + 4)$ sides; prove that the angles

formed at the points of concourse of these sides produced are together equal to $(2n)$ right angles.

3. If a body moves in a curve round a centre of force, and the force by which it is retained in the curve vary in a less ratio than the inverse cube of the distance, prove that the body cannot fall into the centre.

4. Shew that in the spiral, where the angle described by the radius vector $SP \propto SP$ directly; the number of revolutions which have been made by it varies as the square root of the subtangent to the point P.

5. Transform the equation $2x^3 - 2x^2 + 3x + 6 = 0$ into one which shall have its signs alternately positive and negative.

6. Two bodies A and B move in opposite directions with velocities, the sum of which is given. Shew that the sum of the products of each body into the square of its velocity is a *minimum*, when the velocities are reciprocally proportional to the quantities of matter in the bodies.

7. If from one extremity of the diameter of a circle, chords are drawn intersecting the radius which is perpendicular to the diameter or that radius produced, and from the points of intersection ordinates be erected, always equal to the cosine of the arc measured from the opposite extremity of the diameter to the chord; determine the nature of the curve which is the locus of the ordinates.

8. Find the fluent of $\dfrac{dz^{\frac{1}{3}}\dot{z}}{(c^{\frac{5}{3}} - az^{\frac{2}{3}})^{\frac{3}{5}}}$, and of $\dot{x}\int\dot{x}\int\dot{x}\int\dot{x}$ in infinitum; and find the relation of the abscissa and ordinate of a curve when $\varepsilon^z = \varepsilon^x - \varepsilon^{-x}$, ε being the base of hyp. log., and z and x the arc and abscissa respectively.

9. When parallel rays are incident upon a spherical reflector, shew that the radius of the least circle of aberration varies directly as the cube of the semi-aperture, and inversely as the square of the focal length of the reflector.

10. A particle P is attracted to a sphere by forces $\propto \frac{1}{D^2}$. If on the line joining P and the centre of the sphere a semicircle be described and made to revolve; it would cut out a portion of the sphere, the attraction to which is equal to the attraction of the remaining part.

11. Find the sum of the series $\cos. A + \frac{1}{2}\cos. 2A + \frac{1}{3}\cos. 3A + \frac{1}{4}\cos. 4A +$ &c. and resolve $1 + 5 + 19 + 65 + 211 +$ &c. whose scale of relation is $f - g$, into two geometric series whose corresponding terms added together will give the proposed series.

12. Prove that the mean tides are equally affected by the northerly and southerly declination of the Moon.

13. The quadrant of a circle is impelled by a stream in its own plane in the direction of the extreme radius. Find the direction in which it will begin to move.

14. Find the equation to the section of a solid generated by the revolution of a given algebraical curve about its axis.

Tuesday Afternoon.—Mr. Bland.

Fifth and Sixth Classes.

1. Prove that the opposite sides of an equilateral and equiangular hexagon are parallel.

2. Determine the roots of the equation $x^4 - 4x^3 . \sqrt[4]{(2)} + 6x^2 . \sqrt[4]{(4)} - 4x . \sqrt[4]{(8)} + 2 = 0$, and find the equation whose roots are $\frac{1}{2}a + \sqrt{(-\frac{3}{4}a^2)}$; $\frac{1}{2}a - \sqrt{(-\frac{3}{4}a^2)}$; $-a$.

3. If two equal forces, inclined at any angle, act upon a body, prove that the compound force bisects that angle.

4. Given the meridian altitudes of the upper and lower limb of the Sun 18°. 39′. 39″. and 18°. 7′. 3″. Determine its diameter, and the altitude of its centre.

5. Find the sum of the series,

$6 + 2 - 2 - 6 -$ &c. to 19 terms;

$8 + 20 + 50 + 125$ to 15 terms;

$\frac{1}{1 . 2} + \frac{1}{2 . 3} + \frac{1}{3 . 4} +$ &c. to n terms.

6. If a body revolves in a circular orbit about the earth at a distance from its surface equal to 20 radii of the earth; what is the measure of the subtense of the arc described in 1″?

7. If from the extremity of the diameter of a circle a straight line be drawn touching the circle and equal in length to the circumference, and a triangle be formed by joining its extremity and the centre. Prove, that if from any point in this line a perpendicular be drawn to the base, the circumference of the circle described with this as radius, shall be equal to the part of the base intercepted between the perpendicular and the acute angle.

8. The equation $x^4 - \frac{1}{2}x + \frac{3}{16} = 0$ has two equal roots. Find all the roots.

9. Prove that the periodic times of bodies revolving in different ellipses round different centres of force in the foci are in the sesquiplicate ratio of their major axes directly and the subduplicate ratio of the forces inversely.

10. Construct Newton's telescope; and find the magnifying power, and greatest field of view.

11. Find the fluents of

$$\frac{\dot{x}}{x^2\sqrt{(x^2-a^2)}} \quad \frac{\dot{x}}{x^3-2ax^2+x},$$

where a is less than unity: and the n^{th} fluxion of $\sqrt{(x)}$.

12. If a body move in a conic section acted upon by a force tending to the focus S, shew that the velocity at the distance SP is to the velocity at any other distance SQ as a mean proportional between HP and SQ is to a mean proportional between SP and HQ, H being the other focus.

13. Determine the arc of a given circle, the rectangle under whose sine and excess of sine above the cosine is a maximum.

14. The radius of a circle whose area is equal to the surface of a given cone is a mean proportional between the side of the cone and the radius of its base. Required a proof.

15. Compare the absolute forces in the centre and circumference of a circle, so that the periodic times may be the same.

16. An (n^{th}) part of a hollow paraboloid with its vertex downwards is filled with a fluid of known specific gravity; and a sphere of given size and substance is immersed. Find how high the fluid will rise.

TUESDAY AFTERNOON.—MR. HUSTLER.

Third and Fourth Classes.

1. If the two sides of a spherical triangle together $= 180°$, the arc which bisects the vertical angle, bisects the base also.

2. One root of the equation $x^4 + x^3 - 8x^2 - 16x - 8 = 0$ is $1 - \sqrt{(5)}$. Find the other roots.

3. Two bodies are projected towards each other in the same vertical plane from two given points, so as to describe the same parabola. Find the point where they meet.

4. Given the right ascension and declination of a Star and the latitude of the place, determine the day of the year when the Star rises the same instant with the Sun.

5. In the parabola, the normal is the least line which can be drawn from a given point in the axis to the curve.

6. Required the fluxion of $x \times \varepsilon^{\tan. x}$, and the following fluents,

$$\int\frac{\dot{x}}{x^3\sqrt{(1-x^2)}}, \quad \int\frac{bx^{\frac{5}{2}}\dot{x}}{\sqrt{(a^7-x^7)}}, \quad \int\frac{\dot{x}\,\text{h.l.}\,x}{x}.$$

Shew also that $\int\dot{x}\,\text{cosec.}\,2x = \frac{1}{2}\,\text{h. l. tan.}\,x$.

7. If the bases of a cylinder, and of a cone, have the same radius as a sphere, and each of their altitudes = the diameter of the sphere, the solidity of the cone = the excess of the cylinder above the sphere.

8. A body revolving in an ellipse at the mean distance, is projected perpendicularly to the distance with the velocity with which it is there moving. Shew that it describes a circle, and in the same periodic time in which it would have described the ellipse.

9. The equation to a curve is $y^m = ax^{m-1} + x^m$; draw its asymptote, and determine the angle which it makes with the axis.

10. Compare the time of emptying a cone and its circumscribing cylinder, the vertex of the cone being downwards and coinciding with the orifice.

11. If an object be seen through a double convex lens, determine the proportion in which it is magnified, when the distances of the eye and object from the centre are each equal to half the focal length.

12. Sum the following series,

$\frac{1}{1 \cdot 5} + \frac{1}{3 \cdot 7} + \frac{1}{5 \cdot 9}$ to n terms and ad inf.

$1 \cdot 4 + 2 \cdot 5 + 4 \cdot 6 +$ &c. to n terms,

$\frac{1}{1 \cdot 2} - \frac{2}{2 \cdot 3} + \frac{3}{3 \cdot 4} -$ &c. ad inf.

13. Suppose two bodies fall towards a center of force, one acted upon by a force varying as the distance, and the other by a constant force which is half the variable force at the beginning of the motion. Shew that the velocities acquired at the centre will be equal.

14. Give Cotes's construction of the elliptic spiral, and shew in what cases it cuts itself.

15. In the ordinate PN of an ellipse, whose center is C, a point Q is taken so that CQ always $= PN$. What is the curve which passes through the points Q?

16. If a body describes an epicycloid, the force tending to the centre of the globe, required the law of the force.

17. If p and q be two weights applied at the circumferences of a wheel and axle, find the proportion between the radii, so that the time of q ascending through a given space may be a minimum, the inertia of the wheel and axle being considered.

TUESDAY EVENING.—MR. HUSTLER.

1. A pays P £. to B, on condition that he receives an annuity during the life of an individual, who, according to the

tables, may be expected to live n years. What must be the annuity?

2. In an unlimited problem, $mx + ny = p$; if m and n both measure p, m is the least integral value of y, and n of x.

3. A cylinder of indefinite length is placed before a convex reflector, and their axes coincide. Shew that its image is a cone, whose vertex is the principal focus of the reflector,

4. Determine the situation of a fixed Star, so that its right ascension may be unaffected by the precession of the equinoxes.

5. Investigate Taylor's Theorem, by the method of differences.

6. A sphere of given radius is suspended in the air. At a given place, day, and hour, determine the figure of its shadow on a horizontal plane; and shew that the length of the shadow varies as the secant of the Sun's zenith distance.

7. Shew that the variation of the radius of curvature of any meridian of the Earth, is as the square of the sine of latitude.

8. Prove that the value of a vanishing fraction $\frac{P}{Q}$ may be found by taking successively, if necessary, the first, second, and third, &c. fluxions of the numerator and the denominator; and investigate the method of finding the value of such a fraction when the indices are fractional.

9. TB, BC are the subtangent and ordinate of a curve whose vertex is A, and the tangent of the angle TCA is to the tangent of the angle ACB in a given ratio. What is the nature of the curve?

10. For any position of the line of the nodes, construct for the inclination of the Moon's orbit to the plane of the ecliptic. (Newton, Vol. III. Prop. xxxv.)

11. $ABCD$ is a section of a four-sided glass prism perpendicular to its axis, having one angle $D = 90°$, and the opposite angle $B = 135°$. Shew that a ray of light entering the prism perpendicularly to AD, and reflected by AB, BC, will pass through CD without refraction, and thence explain the *Camera lucida*.

12. A paraboloid has its axis parallel to the horizon, and a flexible chain is wound round its greatest circular section; find the length which will be unwound after t'' have elapsed, a given part being unwound at the beginning.

13. Two equal distances CA, CB are drawn at right angles to each other from a centre of force C. In CA any point D is taken, and DE is drawn to CB so that DE may equal CB or CA. Two perfectly elastic balls fall from A and B at the same time, the force varying as the distance, and are reflected at D and E by two planes inclined to their motion at $\angle 45°$. Investigate their subsequent motions.

14. Find the fluents of $\frac{\dot{x}}{x^5\sqrt{(a^2-x^2)}}$, of $\frac{\dot{x}\sqrt{(1+x)}}{(1-x)^{\frac{3}{2}}}$, and of $\frac{\dot{x}\,.\,(1-x^2)}{(1+x^2)\sqrt{(1+ax^2+x^4)}}$.

15. Find the relation between x and y when $\ddot{y}\sqrt{(ay)}=\dot{x}^2$, and shew that $\int\frac{x^{p-1}\dot{x}}{\sqrt{\{(1-x^n)^{n-q}\}}}=\int\frac{x^{q-1}\dot{x}}{\sqrt{\{(1-x^n)^{n-p}\}}}$. between the values of $x=0$, and $x=1$.

16. ACB is a quadrant, and one extremity D of a line CD which equals its chord AB, moves along the radius OB produced, while the other extremity C moves in the periphery of the quadrant. Find the equation and area of the curve described by a point P in the middle of the line CD.

17. If m be any prime number, and x any other number prime to m. Then x and x^m being severally divided by m, leave the same remainder.

18. Sum the series,

$$1+\frac{2}{2\cdot3}+\frac{2^2}{3\cdot3^3}+\frac{2^3}{4\cdot3^3}+\text{\&c. ad inf.}$$

$$\frac{5}{1\cdot2\cdot3\cdot4}+\frac{9}{3\cdot4\cdot5\cdot6}+\frac{13}{5\cdot6\cdot7\cdot8}+\text{\&c. to } n \text{ terms,}$$

and ad inf. by the method of increments.

19. A paraboloid floats in a fluid, the axis not being perpencicular to the horizon. Determine the position in which it rests; the specific gravities of the paraboloid and the fluid being given.

20. CP, CD, are two semi-conjugate diameters of an ellipse, whose center is C. EP, which is perpendicular to CD, is produced to L making PL equal to CD, and through K the middle point in CL, $MKPN$ is drawn so that KM and KN may each equal CK. Shew that the semi-axes major and minor equal PM and PN respectively, and determine their positions.

21. A person turning up three cards from a common pack, undertakes that the number of points upon them shall be either 29, 19, or 9, reckoning 11 or 1 for the ace, and 10 for each of the court cards. What are the odds against him?

22. If a body A be attracted towards two centres of force S and T, and be projected in a direction oblique to the plane STA, the solids generated by the motion of the triangle joining S, T, and the body, shall be proportional to the times of description.

23. A and B are two ships at sea; B moves in a given straight line, and A endeavours to overtake B by always moving towards it. Having given the velocities of A and B, investigate the curve traced out by A.

CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the degree of B.A. in January, 1816.

BY THE TWO MODERATORS.

MONDAY, JANUARY 15, 1816.

MONDAY MORNING.—MR. FRENCH.

First and Second Classes.

1. GIVEN the logarithms of 6 and 7, shew how the logarithm of 1767 may be computed.

2. If the digits composing any number, be inverted, the difference between the number, so formed, and the original number, is divisible by nine.

3. In the oblique impact of an imperfectly elastic body upon a plane, co-tan. incidence : co-tan. reflection : : force of compression : force of elasticity.

4. In Gregory's telescope, the aberrations, produced by the two reflections, are in the same direction.

5. At a given place, on a given day and hour, the Sun's azimuth is double that of a known star; required the distance of the Sun from the star.

6. An imperfectly elastic ball being projected from P, a point in the periphery of a circle PQR, whose centre is C, after impinging at Q and R returns to P; required the value of the angle CPQ.

7. Investigate an expression for the length of a caustic by reflection, and apply it, in the case of parallel rays incident upon a concave spherical reflector.

8. In the common pump, given the height of the fixed sucker above the surface of the reservoir, and the space through which the piston descends; required the altitude of the water in the tube after n strokes of the piston.

9. To determine the radius of curvature in the curve, whose ordinate is equal to the circular arc, of which its abscissa is the versed sine.

10. To find the sum of all the powers of the roots of an equation.

11. To draw a diameter to a curve of n dimensions. (M'Laurin's *Algebra.*)

12. Compare the curvatures of the Moon's orbit in quadrature and syzygy, supposing that the orbit, independently of the Sun's disturbing force, would have been a circle.

13. To determine the number of given points, through which a curve line of the m^{th} order, may be drawn.

MONDAY AFTERNOON.—MR. FRENCH.

Fifth and Sixth Classes.

1. What decimal of a pound is $11\overset{d}{\frac{1}{2}}\frac{2}{3}$?

2. Investigate the rule for finding the least common multiple of two quantities, and apply it to find the least common multiple of 177 and 2982.

3. Required to express the sum of the alternate terms of a binomial raised to the m^{th} power, beginning with the second.

4. If one side of a triangle be bisected, the sum of the squares of the other two sides is double of the square of half the line bisected, and of the square of the line drawn from the point of bisection to the opposite angle.

5. Compare the area of the hexagon inscribed in a given circle with the area of the circumscribing hexagon.

6. Given the figure of an ellipse, find practically its centre and its foci.

7. In an ellipse, if the line $I\,o\,i$ be drawn parallel to the axis minor $B\,C\,D$, and $Q\,o\,q$, parallel to the axis major ACM; then $I\,o \times o\,i : Q\,o \times o\,q :: B\,C^2 : A\,C^2$. Required a proof.

8. Prove, geometrically, that in any plane triangle, the sum of the sides is to their difference as the tangent of half the sum of the angles at the base to the tangent of half their difference.

9. Shew that $\tan.^3 60 = 3 \tan. 60$ to rad. $= 1$.

10. P and W being in equilibrio on an inclined plane, if the whole be put in motion, then P's velocity : W's velocity :: $W : P$.

11. A perfectly hard body is let fall from a given height (a) upon a half-elastic horizontal plane; required the height to which it will rise after impinging the third time upon the plane.

12. Compare the time down $\frac{1}{n}$th part of a given inclined plane with the time down the remainder.

13. Explain the principle of the syphon.

14. Determine the visual angle in Cassegrain's telescope.

15. Given the latitude of the place, and the altitude of the Sun in the equinoctial, find the hour-angle.

16. The force tending to the focus of an hyperbola varies inversely as the square of the distance.

17. Required to express the fluxion of the arc in terms of the fluxion of the co-secant.

18. The roots of the equation $ay^3-by^2-cy+1=0$, are in harmonic progression, find them.

19. Required the fluents of $\frac{mbx^{-n-1}\dot{x}}{\sqrt{(e+fx^{-n})}}$, $\frac{\sqrt{(y)}\cdot\dot{y}}{1+y^{\frac{3}{2}}}$,

$$\frac{\dot{x}}{\sqrt{(x^2-a^2)}},\quad \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{(2a-x)}},\quad \frac{z\dot{z}}{\sqrt{(a^4-z^4)}}.$$

MONDAY AFTERNOON.—MR. BLAND,

Third and Fourth Classes.

1. Prove that the sectors of two different circles are equal, when their angles are inversely as the squares of the radii.

2. If a circle be described about the centre of gravity of a system of bodies A, B, C, &c., and any point S be taken in the circumference, shew that $A\times SA^2+B\times SB^2+C\times SC^2+$&c. is constant.

3. The radius of curvature of the common parabola has to the normal the duplicate ratio that the normal has to the semi-parameter. Required proof.

4. Find the centre of gravity of the solid generated by a quadrant of a circle through one-fourth of a revolution about the radius.

5. Given (a) and (b) the $(m)^{th}$ and $(n)^{th}$ terms of an arithmetic progression; determine the value of the $(z)^{th}$ term.

6. Find the fluents of

$$\frac{d\dot{x}}{a^3-mx^2},\quad \frac{d\dot{x}\sqrt{x}}{\sqrt{(1-x)}} \text{ and } \frac{x^2\dot{x}}{(x-a)^2\cdot(x+a)}.$$

7. From a given right cone cut off a parabola such, that its area shall be double the rectangle contained by the segments of the diameter of the base, formed by the section.

8. If the force $\propto\frac{1}{D^2}$, and a body be projected in a given direction with a velocity which is to the velocity in a circle at the same distance in a less ratio than $\sqrt{2}:1$; determine the nature of the orbit described.

9. If an object be placed between two plane reflectors inclined to each other at any angle, and the eye of a spectator be in any

point between the planes, the distance of the eye from any of the images seen by him, is equal to the length of the path described by the rays which form that image.

10. Prove that no fraction can be represented by a terminating decimal, unless the denominator be 2 or 5, or the product of some powers of 2 and 5.

11. The angle included between the hour-lines of 12 and 3, is equal to the angle included between the hour-lines of 4 and 6, in a horizontal dial. Determine the latitude for which the dial is constructed.

12. What must be the nature of a parabolic curve which revolving round its axis would generate a solid, such that the time of emptying it would be to the time of emptying the circumscribing cylinder in the ratio of 1 : 9.

13. If the attraction of the earth and moon be as their quantities of matter directly and the squares of their distances inversely; what is the nature of the curve in which a body being placed would be equally attracted to both.

14. Compare the ablatitious force with the mean force of P to T (11 Sect.); the periodic times of P and T being as $1 : n$.

15. Prove that if two bodies be projected in similar directions with velocities which are in the subduplicate ratio of the force and distance, they will proceed in similar curves.

MONDAY EVENING.—MR. BLAND.

1. Given the sum (s) and the sum of the squares (S) of a geometric series continued in infinitum. Determine the series.

2. If the bases of two equal cycloids be parallel, and the vertex of one in the base of the other; prove that the angle formed by the intersection of the curves will be a right angle.

3. If two conic sections be described on the same axis major, and have the same abscissæ, the ordinates will be in the subduplicate ratio of the *latera recta*. Required proof.

4. In a system of wheels moveable by teeth and pinions, having given the ratios of the number of teeth in each wheel and pinion, determine the number of times the $(n)^{\text{th}}$ wheel turns round its axis, while the first performs (m) revolutions.

5. Trace the curve whose equation is $a \cdot (y-b)^2 = x \cdot (x-a)^2$, and determine the angle at which the curve cuts the axis when $x = a$.

6. If the altitudes of the Sun be taken at the same place on the same day, when he is in the same vertical in opposite directions; shew that the sum of their tangents will be to the sum of their secants, as the sine of the Sun's declination is to the sine of the latitude of the place.

7. Prove that the sums of the reciprocals of the $(n)^{th}$ powers of the odd and even numbers are to each other in the proportion of $2^n - 1 : 1$.

8. Find the sum of the series: $\frac{2a+b}{a^2.(a+b)^2} + \frac{2a+3b}{(a+b)^2.(a+2b)^2} + \frac{2a+5b}{(a+2b)^2.(a+3b)^2} +$ &c. in inf.

$\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} +$ &c. to n terms by the method of increments.

and $\frac{1}{4.7.12} + \frac{1}{6.10.16} + \frac{1}{8.13.20} +$ &c. in inf.

9. Determine the inclination of a plane of given length, so that a cylinder of known dimensions and uniform density, may roll freely down it in a given time.

10. A barometer having some air in the tube, stands at an altitude of (a) inches; but being put under the receiver of an air-pump which contains (n) times as much as its barrel, after m turns it stands at an altitude (b). Find the standard altitude, and the quantity of air in the tube at first.

11. An aperture of given area is cut from the top to the bottom of the side of a regular vessel full of water. Required its nature and dimensions, such that the velocity of the descending surface may be as the $(n)^{th}$ power of its distance from the lowest point; the velocity of every particle of the issuing fluid being supposed to vary as the square root of its depth below the surface.

12. Shew that in the spiral where the angle described by the radius vector SP $\propto$ SP^m the areas described by SP in one, two, three, ... n revolutions, measuring from the centre, will be as the numbers, 1, 2, 3 .... n raised to the power $\frac{m+2}{m}$.

13. Find the fluents of $\frac{(a+bx).d\dot{x}}{x^3-1.}$, $\frac{\dot{x}}{x^n\sqrt{(a^2-x^2)}}$ where n is an even number, and $\frac{d\dot{x}}{x.(1+x)^2.(1+x+x^2)}$.

14. Two imperfectly elastic bodies A and B are at a given distance in the same vertical line: A, the higher is acted on by gravity, which is supposed to have no effect on B. Shew that if A falls and strike B successively, the intervals between the strokes decrease in geometric progression; and determine the space passed over after any number of strokes.

15. If a small pencil of diverging rays be incident, nearly

perpendicularly on the spherical surface of a plane convex glass mirror, the radius of which is known; determine what must be its thickness, so that after refraction at the convex, and reflection at the plane surface, they may converge to that point where the axis intersects the spherical surface; the focus of incidence being given, and at a greater distance from the surface than the diameter of the sphere.

16. Determine the nature of the curve which will refract parallel rays to or from one focus, when the cosine of incidence is to the sine of refraction :: $1+n : 1$.

17. Given the greatest and least apparent diameters of the Moon, find what would be the apparent diameter corresponding to the mean distance; and shew that it is less than the mean apparent diameter.

18. If any number of bodies be retained in horizontal circular orbits by means of strings of unequal lengths, and the distances of the centres from the points of suspension be equal, the times of their revolutions will be the same.

19. Determine the nature and area of a curve such, that if a right line be drawn from its vertex making an angle of 45° with the axis, the portion of the ordinate intercepted between this line and the curve shall always have to the sub-tangent the inverse ratio that the ordinate has to the given line (a).

20. Let a sphere of a given diameter be projected in a fluid, the specific gravity of which is to that of the sphere as $1 : n$; having given the velocity of projection, determine what part of it is lost during the time the body describes any given space.

21. Shew that the effect of the Sun upon the matter of the Earth exterior to the inscribed sphere, to turn it about its centre, is equal to the effect which would be produced if one-fifth part of that matter was placed at that point of the Earth's equator which is opposite to the Sun.

22. Required the area of the rhumb-line considered as a spiral; and shew that its orthographic projection on the plane of the equator is an hyperbolic spiral.

23. If the particles of air be moved from their places by a force which varies according to any given law; it is required to find the law of the force with which they will *continue* to be agitated, supposing the elasticity of the atmosphere to be proportional to its density.

24. If a chain of given weight reaching to the centre of the Earth be suspended from a cylinder at the surface, round which it is made to wind itself by the descent of a weight (w) unwinding a string supposed to be without weight; determine the velocity of w at any point, and also where it is the greatest.

TUESDAY MORNING.—MR. BLAND.

First and Second Classes.

1. If a series of arcs be taken in arithmetic progression, the radius of the circle will be to twice the cosine of the common difference as the cosine of any arc taken as a mean is to the sum of the cosines of any two equidistant extremes.

2. Find the sum of a recurring decimal $\cdot qppp$ &c. in inf., (q) and (p) containing (m) and (n) digits respectively.

3. In the direct impact of perfectly hard bodies, the difference between the sums of the products of each body into the square of its velocity before and after impact, is equal to the sum of the product of each body into the square of the velocity gained or lost.

4. If S, a, r, n be respectively the sum, first term, common ratio and number of terms of a geometric progression; find the sum of the series,

$$(S + a) + (S + a + a r) + (S + a + a r + a r^2) + \&c.$$

5. Prove that if contiguous and parallel rays of light fall upon a refracting sphere, the homogeneal rays will emerge parallel after n reflections and two refractions, when the least cotemporary variations of the angles of incidence and refraction are to each other as $n + 1 : 1$.

6. If an equilateral triangle be inscribed in a circle, and the adjacent arcs, cut off by two of its sides, be bisected: the line joining the points of bisection will be trisected by the sides.

7. The sum of the distances of a star from two known stars is a minimum, and its declination, which is greater or less than that of each of the others is known; determine its right ascension.

8. Let $y = A + Bx + Cx^2 + Dx^3 +$ &c. where A, B, C, D, &c. are constant quantities; then if $[y]$, $\left[\frac{\dot{y}}{\dot{x}}\right]$, $\left[\frac{\ddot{y}}{\dot{x}^2}\right]$, &c. be the values of y, $\frac{\dot{y}}{\dot{x}}$, $\frac{\ddot{y}}{\dot{x}^2}$, &c. when $x = 0$; Prove that

$$y = [y] + \left[\frac{\dot{y}}{\dot{x}}\right] \cdot \frac{x}{1} + \left[\frac{\ddot{y}}{\dot{x}^2}\right] \cdot \frac{x^2}{1 \cdot 2} + \left[\frac{\dddot{y}}{\dot{x}^3}\right] \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

9. Find the fluent of $\frac{\dot{z}}{z\sqrt{(a+cz^n)}}$, and of $\frac{x^p\dot{x}}{(1+x^n)^2}$, the fluent of $\frac{x^p\dot{x}}{1+x^n}$ being = A.

10. The velocity of a body descending from an infinite distance towards a centre of force, is $\frac{1}{n}$th part of the velocity in a circle at the distance of that point. It is required to determine the law of the force.

11. If P be the place of a comet in its parabolic orbit, and a circle be described through P, the vertex and the focus; shew that the time of moving from perihelion to P will be proportional to the perpendicular drawn from the centre of the circle to the axis.

12. If $ay^m\dot{y} = cx^n\dot{y} - ayx^{n-1}\dot{x}$; determine the algebraic relation of x and y.

13. In latitude 45° the mean altitude of the tide is always the same whatever be the declination of the Moon.

14. From two bags, one of which contains (m) and the other (n) balls, marked a, b, c, d, &c. (m) being greater than (n), two balls are drawn; What is the probability that they have both the same letter?

TUESDAY AFTERNOON.—MR. BLAND.

Fifth and Sixth Classes.

1. Extract the square root of $a + x + \sqrt{(2ax + x^2)}$.

2. The sum of (n) terms of any arithmetic progression whose common difference is equal to the least term, will be equal to the sum of $(n+1)$ magnitudes, each of which is half the greatest term of the progression.

3. If four quantities of the same kind be proportional; the first shall have to the third the same ratio that the second has to the fourth. (EUC. B. 5.)

4. If $A \propto B$, and $B \propto C$; prove that $A \propto mB \pm nC$, where (m) and (n) are known quantities.

5. In the steel-yard, if the weight increase in arithmetic progression, the divisions of the scale will be at equal intervals; and if each of these intervals be equal to the shorter arm, the moveable weight will be equal to the difference of the arithmetic progressionals.

6. Two bodies descend, one vertically through 400 feet, and

the other down an inclined plane 500 feet long, and inclined at an angle of 30° to the horizon, compare the time of their descents.

7. The solid content of a cone whose base is equal to a great circle of a sphere, and altitude equal to the diameter, is half the solid content of the sphere.

8. If the force $\propto \frac{1}{D^2}$, determine how far a body must fall externally to acquire the velocity in the ellipse.

9. Find the fluents of $\frac{(a+bx)\,.\,\dot{x}}{a^2+x^2}$ and $\frac{x^4\dot{x}}{\sqrt{(a^2+x^2)}}$.

10. Construct Newton's telescope, investigate its magnifying power, and find the linear magnitude of the greatest field of view.

11. If a vertical straight line be placed before a plane mirror inclined at an angle of 45° to the horizon, determine the image and its position.

12. Given $\sqrt{\frac{(x+a)^3}{x-a}}$ = a minimum. Find the value of x.

13. A fluid issuing from the side of a vessel (h) feet high, struck the horizontal plane at a distance of (d) feet from the bottom. Determine the point in the side of the vessel where the orifice is made.

14. If the force $\propto \frac{1}{D^2}$, and bodies fall from different altitudes towards different centres of force, determine the proportions of the times in which they fall through any space.

15. If the force $\propto \frac{bA^m + cA^n}{A^3}$; find the angle between the apsids.

16. If A, A′, A″, represent the areas of three similar rectilineal polygons described on the hypothenuse and sides of a right-angled triangle, $A = A' + A''$. Required a proof.

TUESDAY AFTERNOON.—Mr. FRENCH.

Third and Fourth Classes.

1. Prove, geometrically, that $1 + \cos. 2\theta = 2\cos.^2\theta$.

2. A perfectly elastic ball, projected from A directly up an inclined plane AB, strikes a vertical plane passing through B and returns to A; required its velocity at B, the length of the plane being 36 feet, and its elevation 30°.

3. In an hemispheroid emptying itself by a small orifice in

the vertex, compare the time in which the surface of the fluid descends through the upper half of its axis, with the time through the lower.

4. To find the variation of the angle, which a given object subtends at the eye when viewed through a convex lens, the object being farther from the lens than its principal focus (F), and the eye nearer to the lens than its principal focus (f).

5. The precession in right ascension is positive when the angle of position is acute, and negative when it is obtuse. Required a proof.

6. The least error in time due to a given error in the altitude of a known star being b''; to determine the latitude of the place, and the true zenith-distance of the star.

7. To find the area of the Conchoid of Nicomedes.

8. If the moveable orbit be projected *in antecedentia*, with a velocity equal to that of P *in consequentia*, shew that the velocity of p vanishes, when Cp becomes the least distance in the ellipse.

9. Investigate the equation between the perpendicular and the distance, in the lituus $\left(\angle \propto \frac{1}{\text{Dist.}^2}\right)$ and determine the point of contrary flexure in the curve.

10. The roots of the equation, $y^4-8y^3+14y^2+8y-15=0$, are in arithmetical progression. Find them.

11. Required the fluxion of the arc whose sine $=2y\sqrt{(1-y^2)}$.

12. Apply Newton's method of making a body oscillate in a cycloid, to the common cycloid.

13. Find the following fluents:

$$\int\frac{x^4\dot{x}}{\sqrt{(a^2-x^2)}},\quad \int\frac{2a\dot{x}}{x\sqrt{(a^3+x^3)}},\quad f.\ \dot{\theta}\cos.^3\theta.$$

14. Sum the following series:

$$\frac{1}{1\cdot3}-\frac{1}{3\cdot5}\ \text{\&c. in inf.}$$

$$1+3+7+15+\text{\&c. to } n \text{ terms.}$$

$$\frac{1}{1\cdot5\cdot9}+\frac{1}{5\cdot9\cdot13}+\text{\&c. to } n \text{ terms by increments.}$$

TUESDAY EVENING.—MR. FRENCH.

1. If a sum p, at compound interest, in n years amounts to m, in what time will the same sum amount to M, at the same rate?

2. A cylindrical vessel, of given thickness, is required to be of a certain capacity, find the least quantity of material with which it can be made.

3. A thin rod, formed of the arcs of two entire, unequal cycloids, lying in the same plane, and on opposite sides of the line of their bases, floats upon a fluid, sinking to the point at which the arcs are united; to determine its position when at rest.

4. To find the surface of a sphere by the method of indivisibles.

5. Required the time of oscillation in a finite circular arc.

6. To determine that point in the periphery of an ellipse, at which the angle contained between the normal and the distance from the centre, is a maximum.

7. The place of the Earth, when a Star's aberration in declination $= 0$, being d_0, and A_0 (lying to the eastward of the point of syzygy) being the place of the aberratic point, when its aberration in right ascension $= 0$; required the position of d_0 with respect to A_0.

8. Solve the following equation $x^3 - 9x + 28 = 0$ by a process similar to that employed by Cardan.

9. If a small pencil of parallel rays fall upon a concave spherical surface, and every ray be reflected, the density of the incident pencil is to the density (supposed uniform) of rays in the least circle of aberration, as the area of the circle, whose diameter is the versed sine of the aperture, to the whole surface of the sphere, very nearly; Required a proof.

10. Find the n roots of unity, and shew how the quadratics combine, when n is an odd number.

11. The weight of a given globe being inconsiderable, when compared with the weight of an equal bulk of fluid, prove that, in its ascent, the velocity is uniform, and equal to the velocity by gravity through $\frac{4}{3}$ds of its diameter.

12. Investigate an expression for the pressure on the axis of a mechanical power in motion, and apply it in the case of the single fixed pulley.

13. Sum the following series:

$$\frac{11}{1\cdot2\cdot3\cdot4} + \frac{17}{4\cdot5\cdot6\cdot7} + \frac{23}{7\cdot8\cdot9\cdot10} + \text{\&c. to } n \text{ terms.}$$

$$\frac{5}{1\cdot2\cdot3} \times \frac{1}{2^2} + \frac{6}{2\cdot3\cdot4} \times \frac{1}{2^3} + \text{\&c. to } n \text{ terms.}$$

$$\frac{1}{1\cdot2} - \frac{1}{4\cdot5} + \frac{1}{7\cdot8} - \text{\&c. in infinitum.}$$

14. Explain D'Alembert's *principle;* and apply it to find the accelerating force on a body drawn up an inclined plane, by the action of a power parallel to the plane.

15. In any square number, 4 is the only digit which can occupy both the units and tens places.

16. To find the whole number of equal and regular figures, which may be described upon the surface of a sphere so as exactly to cover it.

17. BM is a chord of a circle, whose centre is C, and CEF any radius cutting BM in the point E; at every point E, EP is erected perpendicular to BM and equal to EF; required the locus of P.

18. In the catenary, the horizontal tension is the same at every point, to determine its actual value.

19. Find the following fluents:

$$\int \frac{x\dot{x}}{(1+x)^3 \times (1+x+x^2)^{\frac{1}{2}}}, \quad \int b \times x^{\frac{3}{2}}\dot{x} \times \frac{\sqrt{(2a-x)}}{(a-x)^2},$$

$$\int az^2 \times y^{n-2}\dot{y} \text{ if } \dot{z} = (b + cy^n)^m \times \dot{y}.$$

20. A body, acted upon by gravity, is projected horizontally, with a given velocity, along the interior surface of a cylinder; required to trace its path upon the surface of the cylinder.

21. To calculate the probability of throwing two assigned numbers, A and B. with m dice, in n throws.

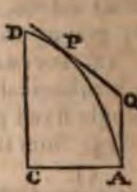
22. Solve the following fluxional equations:

$$(x-y) \cdot \dot{x}\dot{y} = y \cdot \dot{x}^2 + (a-x) \cdot \dot{y}^2$$

and $\frac{\ddot{x}}{\dot{z}^2} + x + \cos. mz = 0$.

23. If an equilateral polygon of 2^n sides be inscribed in a circle, whose rad. $= 1$, the value of each side is $\sqrt{[2 - \sqrt{(2 + \sqrt{(2 + \&c.)})}]}$ where the numeral 2 is repeated $(n-1)$ times.

24. ADC is a common parabola, AQ a tangent at the vertex, and PQ a tangent at P, meeting it in Q; to determine the point P, such, that the resistance on the solid, generated by the revolution of DPQA about CA, when moving in the direction of its axis, may be the least possible.



CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the degrees of B. A. in January, 1817.

BY THE TWO MODERATORS.

MONDAY, JANUARY 13, 1817.

MONDAY MORNING.—MR. PEACOCK.

First and Second Classes.

1. What decimal of £1. is 3*s*. 3¼*d*. ?

2. Find the integer values of x and y, which satisfy the equation $13x + 14y = 200$

3. Prove that $\theta = \tan.\,\theta - \frac{1}{3}\tan.\,\theta^3 + \frac{1}{5}\tan\theta^5 - \&c.$

4. Find the integral of $\frac{dx}{1 + x^3}$.

5. Explain what is meant by the *particular solutions* of differential equations. Give an instance in the equation

$$y\,dx - x\,dy = n\surd\left\{dx^2 + dy^2\right\}$$

6. Find expressions for the range and time of flight of a body projected from a given point *above* a given plane.

7. Two vessels filled with air of different densities communicate by a tube. Find the velocity with which the air will rush into the vessel containing the rarer air.

8. Explain the causes of the following lunar inequalities:

(1.) The evection.
(2.) The variation.
(3.) The annual equation.

9. The aberration which arises from the spherical surfaces of lenses is very small, compared with that which is caused by the unequal refrangibility of light.

10. Prove that in the Calculus of Variations,

$$\delta d v = d \delta v,$$

v being a function of x.

MONDAY AFTERNOON.—MR. PEACOCK.

Fifth and Sixth Classes.

1. Find the quotient of 75.04 divided by 3.02101 to three places of decimals.

2. Find the amount of £70. in three years, at $3\frac{1}{2}$ per cent. allowing simple interest.

3. Shew that

$$\left\{\sqrt{(a+b\sqrt{(-1)})}+\sqrt{(a-b\sqrt{(-1)})}\right\}^2$$
$$=2a+2\sqrt{\left\{a^2+b^2\right\}}.$$

4 The number N is divisible by 7, if $a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3 + \&c. + a_n \cdot 3^n$, be divisible by 7; $a_0, a_1, a_2,$ &c. being the digits of the number reckoning from the place of units.

5. The middle term of the expansion of $(1+x)^n$, when n is even, is

$$= 2^{\frac{n}{2}} \cdot \frac{1 \cdot 3 \cdot 5 \ldots\ldots (n-1)}{1 \cdot 2 \cdot 3 \ldots\ldots \left(\frac{n}{2}\right)} \cdot x^{\frac{n}{2}}$$

6. Explain the method of constructing a table of sines.

7. Solve the equation

$$x^4 + px^3 + qx^2 + px + 1 = 0.$$

8. There are at least as many impossible roots in the original equation, as in the equation of limits.

9. Explain what is meant by the modulus of a system of logarithms and shew how it is determined.

10. If $\pi = 3.14159$, t = time of oscillation, and l the length of pendulum, then

$$t = \pi \sqrt{\frac{l}{2\,mf}}.$$

11. Explain the method of determining the right ascensions of the fixed stars.

12. Enumerate and explain the phænomena exhibited by the moon in the course of a month.

13. Explain the method of determining the specific gravities of bodies.

14. A given rectilinear object is placed at a given point in the axis of a concave mirror. Required the nature and position of its image.

15. Find the fluxion of $\sin x$.

16. Find the values of x, which make the function $x^3 - x^2 - 8x + 12$ a *maximum* or a *minimum*.

17. Expand $(1+x)^{\frac{1}{2}}$ by means of Maclaurin's theorem.

MONDAY AFTERNOON.—MR. WHITE.

Third and Fourth Classes.

1. A sum P is due at the end of m years; find the difference between its amount at the end of $(m+n)$ years, and the amount of its present value at the end of $(m+n)$ years, at simple interest.

2. It is required to find two harmonic means between 3 and 12.

3. If two circles intersect each other in A, B; any two parallel lines CD, EF, drawn through A, B, respectively, and cutting the circles in C, D; E, F; are equal. Required a proof.

4. An object is placed before a concave spherical reflector. Required its position, when the image is inverted, and equal to twice the object.

5. It is required to find the principal focus of a concavo-convex lens, of a rarer medium, whose thickness is inconsiderable.

6. A body falls down a given length of an inclined plane, and impinging upon the horizontal plane moves along it; required the elevation of the plane, when the time of moving upon the horizontal plane, over a space equal to the height of the plane, is equal to the time down the plane.

7. Prove that when a body oscillates in a cycloid, the whole force which stretches the string, varies as the length of that part of it which is not in contact with the upper cycloid.

8. The length of the shadow of an upright rod at noon on the shortest day : its length at noon on the longest day :: n : 1. Prove that the sine of twice the latitude : the sine of twice the obliquity :: $(n+1) : (n-1)$.

9. A hemispherical vessel standing upon its base is filled with fluid; compare the pressures perpendicular to its plane and curved surfaces.

10. Compare the times of emptying a vessel in the form of a parabolic frustum, by a small orifice in its base, when it is placed with the vertex of the parabola downwards, and when it is placed with the vertex upwards.

11. Compare the time of descent in a given logarithmic spiral, to the centre S, from a given point P, with the periodic time in a circle, at the distance SP.

12. If a body fall down the radius of a circle, F varying as $(\text{dist.})^3$, and ascend on the other side through radius by a repulsive force; shew that it will acquire the velocity of revolution in the circle.

13. In any spherical triangle whose sides are a, b, c, and opposite angles A, B, C; if $b = c$, shew that

$$\text{sin. } b = \frac{\text{sin.}\frac{a}{2}}{\text{sin.}\frac{A}{2}}, \text{ and sin. } B = \frac{\text{cos.}\frac{A}{2}}{\text{cos.}\frac{a}{2}}.$$

14. Sum the series

$$\frac{3}{1 \cdot 2 \cdot 2} + \frac{4}{2 \cdot 3 \cdot 2^2} + \frac{5}{3 \cdot 4 \cdot 2^2} + \text{\&c. to } n \text{ terms.}$$

$$\text{and } \frac{1}{1 \cdot 5} + \frac{1}{3 \cdot 7} + \frac{1}{5 \cdot 9} + \text{\&c. to } n \text{ terms.}$$

15. Find the fluents of

$$\frac{p\dot{x}}{\sqrt{(x)} \cdot \sqrt{(a - bx)}}, \text{ and } \frac{\dot{x}}{x\sqrt{(1 + \sqrt{(x)})}}.$$

16. Find the centre of gyration of the plane of a semicircle, revolving about its diameter.

MONDAY EVENING.—MR. WHITE.

1. The present value of a freehold estate of £100. *per annum*, subject to the payment of a certain sum (A) at the end of every two years, is £1000. allowing 5 *per cent.* compound interest. Required to determine the sum A.

2. $1^3 = 1$, $2^3 = 3 + 5$. $3^3 = 7 + 9 + 11$, $4^3 = 13 + 15 + 17 + 19$, &c. = &c. Prove the formula for n^3.

3. Two equal hard bodies are projected at the same instant towards each other, from the two extremities of a vertical line, each with the velocity which would be required by falling down it. Required the interval of time, between their impact and their arrival at the lower extremity of the line.

4. A hemispherical vessel, of given weight, floats upon a fluid, with one third of its axis below the surface. Required the weight which must be put into it, so that it may float with two-thirds of its axis below the surface.

5. A ray of light passes through a prism of a denser medium, and the ray within makes two acute angles with the sides of the prism; if I, i, be the angles which the incident and emergent rays make with the perpendiculars to the surfaces, and R, r, the angles which the ray within makes with the same perpendiculars, prove that when the deviation is a minimum, $I = i$, and $R = r$.

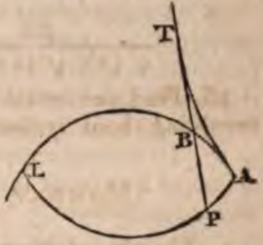
6. If the Moon and Sun be upon the meridian at the same instant, and A, a, be the increases of their right-ascensions (sup-

posed uniform) in one solar day; A, a, being reckoned in time at 15° to one hour; shew that the exact interval between their next following transits $= \frac{(A - a)}{24 - (A - a)} \times 24$ hours, solar time.

7. Prove that by means of the series of weights 1, 2, 4, 8, 16, &c. any weight not exceeding the sum of the weights, can be weighed.

8. Prove that a circular arc of given radius, will oscillate through a given angle, in its own plane, about its middle point, in the same time, whatever be its length.

9. When a body oscillates in a hypocycloid, as in the tenth section; if T B P be any position of the string, shew that the time of describing A P : the time of describing P L : : the arc A B : the arc B L.



10. Prove that the roots of the equation, $x^m + 1 = 0$, when m is an odd number, are, $\frac{1}{r^{m-2}} \ldots\ldots \frac{1}{r^3}, \frac{1}{r}, r, r^3, \ldots\ldots$ r^{m-2}, r^m, where $r = \left(\cos.\frac{\pi}{m} + (\sqrt{-1})\sin.\frac{\pi}{m}\right)$ and $r^m = -1$.

11. Prove that a regular octahedron inscribed in a sphere : the cube of the radius : : 4 : 3.

12. A body is projected, at a given distance r, at an angle of 30°, with the velocity acquired from infinity. Find the time elapsed when the body is at the distance $\frac{r}{2}$ from the centre; supposing the force to vary as $\frac{1}{(\text{dist.})^2}$, and to be equal to twice the force of gravity at the point of projection. (Newt. Sect. 8.)

13. The sides of a spherical triangle are a, b, c; and the opposite angles A, B, C; if A and C be invariable, and b be increased by a small quantity, shew that a will be increased or diminished, according as c is less or greater than a quadrant.

14. The mean motion of the nodes of the fourth satellite of Jupiter, caused by the disturbing action of the third, ought, ac-

cording to the principles of the eleventh section, to be regressive; whilst this regression takes place, can the node of the orbit of the fourth satellite be progressive upon Jupiter's orbit?

15. When rays diverge, from a point beyond its principal focus, upon a double convex lens of a denser medium; if q' be the distance of the focus of refracted rays from the second surface, the thickness (t) being small; and q that distance when t is neglected; shew that $\frac{1}{q'} = \frac{1}{q} + \frac{(nd - r)^2}{(1+n)\,d^2 r^2} \cdot t$, nearly; where r is the radius of the first surface, d the distance of the focus of incidence from it, and $1 + n : 1 :: \sin \text{I} : \sin \text{R}$.

16. A ball, whose elasticity : perfect elasticity :: $n : 1$, is projected obliquely upwards, from a point in the horizontal plane, upon which it impinges and rebounds continually. Prove that the ranges and times of flight in the successive parabolas described, form geometric progressions; and find their sum.

17. If the resistance vary as $(\text{vel.})^3$, and a body fall by the action of a constant force; find the time in which it will acquire a given velocity.

18. Find the fluent of $\frac{\dot{x}}{(a^2 + x^2)^{\frac{2r+1}{2}}}$, and in the fluent of $\frac{x^m \dot{x}}{x^n - px^{(n-1)} + qx^{(n-2)} - \&c.}$, m being greater than n, shew that the coefficients of all the terms which involve higher powers of x than the $(m - n + 1)^{th}$ will vanish.

19. Sum the following series,

$$\frac{1}{1 \cdot 3 \cdot 3} - \frac{2}{3 \cdot 5 \cdot 3^2} + \frac{3}{5 \cdot 7 \cdot 3^3} - \&c. \text{ ad infinitum.}$$

also, $\frac{2}{1 \cdot 3 \cdot 3} + \frac{3}{3 \cdot 5 \cdot 3^2} + \frac{4}{5 \cdot 7 \cdot 3^3} + \&c.$ to n terms by the method of increments.

20. An isosceles right-angled triangle is immersed in a fluid, having one of its sides coincident with the surface; find the distance of the centre of pressure from the side immersed.

21. (Sect. XI. Prop. 66.) Cor. 14. If S T and the absolute force of S be changed, the periodic linear errors of P $\propto \frac{1}{(\text{Per}^{c}.\ \text{Time of T})^2}$. Cor. 15. If ST, PT, be changed in any

proportion, and also the absolute forces of S and T be changed in the same proportion, the periodic linear errors of P vary as PT. Required proof: and hence to compare the periodic linear errors of P in different systems of S, T, P, where the form and inclination only of the orbits remain the same.

22. If out of 86 persons born, one dies at the end of every year; and m, n, be the complements to 86 of the ages of two individuals A, B, m being less than n, prove that the probability of A's surviving $B = \frac{m-1}{2n}$.

23. In a system of two pullies, where each string is attached to the weight, P draws up W; find the accelerating force on P, the tensions of the strings, and the pressures upon the centres of the pullies; taking into consideration the weight and inertia of the pullies.

TUESDAY MORNING.—Mr. WHITE.

First and Second Classes.

1. The logarithm of 37852 is 4.5787767, the logarithm of 37853 is 4.5787882. Required the number, corresponding to the logarithm 6.5787836.

2. If the sides of a spherical triangle, AB, AC, be produced to b, c, so that Bb, Cc, shall be the semi-supplements of AB, AC, respectively; prove that the arc bc will subtend an angle at the centre of the sphere, equal to the angle between the chords of AB, AC.

3. If the radii, of the tube, and of the basin, of a barometer, be 1 and 3; and the index shews, at sight, the height of the mercury in the tube, above that in the basin; prove that the inch upon the scale : a real inch :: 8 : 9, the thickness of the tube being neglected.

4. The altitudes of a circum-polar star are observed at two instants, when it has the same azimuth, before it passes the meridian; and also the time between those instants; from these data, determine the latitude of the place.

5. Having given the distance at which a short-sighted person can see distinctly, it is required, to find the distance between a given object-glass, and given eye-glass, in an astronomical telescope, when adapted to such an eye, and to distant objects.

6. The periodic times of the first and second satellites of Jupiter, are, ($1^d. 18^h. 27^m. 33^s.$) and ($3^d. 13^h. 13^m. 42^s.$). If a, a',

be their mean distances, prove that, $a : a' :: 1 : 2^{\frac{2}{3}} \times \left(1 + \frac{2}{3} \times \frac{1}{274}\right)$ nearly.

7. If the $\sqrt[3]{\left(\tan.\left(45 - \frac{z}{2}\right)\right)} = \tan. u.$ prove that $\sqrt[3]{(\tan. z + \sec. z)} + \sqrt[3]{(\tan. z - \sec. z)} = 2 \cot. 2 u.$

8. Determine the weights which must be selected out of the series, 1, 2, 4, 8, &c. pounds, in order to weigh 1317 pounds.

9. If a body be projected obliquely upwards, shew that the square of its velocity, will always be equal to the square of the velocity of projection, diminished by the square of the velocity which it would acquire by falling down its perpendicular height, above the horizontal plane passing through the point of projection.

10. A body describes a circle, the centre of force being in the circumference; another body describes an equal circle, the centre of force being in the centre of the circle, and the absolute force being one fourth of its former value. Compare the times in which the circles are described.

11. Prove, that $(a + b)^n = a^n + n \cdot a^n \cdot \frac{b}{(a + b)} + n \cdot \frac{n + 1}{2} \cdot a^n \cdot \frac{b^2}{(a + b)^2} + n \cdot \frac{n + 1}{2} \cdot \frac{n + 2}{3} \cdot a^n \cdot \frac{b^3}{(a + b)^3}$ + &c.

and, by this Theorem, sum the series,

$\frac{1}{3} + \frac{3}{2} \cdot \frac{1}{3^2} + \frac{3 \cdot 4}{2 \cdot 3} \cdot \frac{1}{3^3} + \frac{3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4} \cdot \frac{1}{3^4}$ + &c. ad infinitum.

12. Upon one side of the given straight line AB describe a semicircle, and upon the other side an equilateral triangle ADB; if a solid be generated by the revolution of this figure about DC, C being the centre of the semicircle; prove that it will rest upon the horizontal plane, upon any point of its spherical surface.

TUESDAY AFTERNOON.—MR. WHITE.

Fifth and Sixth Classes.

1. It is required to express 23°. 27′. 53″. in hours, minutes, and seconds.

2. Find the discount upon £125. 10*s*. 0*d*. payable at the end of three years, at $4\frac{1}{2}$ *per cent*, simple interest.

3. It is required to determine the point C, in the semicircle ACB, such that the three sides of the triangle ACB shall be in geometrical progression.

4. Two bodies 1, 2, moving with velocities 1, 2, whose elasticity : perfect elasticity :: 1 : 2, impinge upon each other, making the angles of 30°, and 90°, respectively, with the plane touching them at the point of contact. Required the directions in which they will move, and their velocities after impact.

5. A body is projected down an inclined plane, with the velocity acquired in falling down its height, and describes the length of the plane in the time of falling down its height. Required the elevation of the plane.

6. In a quadrantal triangle, the angle opposite the quadrant, and one of the other angles, are given; find the remaining angle.

7. Prove that the illumined phase of Mars is the least, when he is in quadrature.

8. If an object be viewed through a glass plate of given thickness, determine how much the apparent distance is less than the true.

9. It is required to determine the brightest part of the visible area in Galileo's telescope.

10. A circle and its inscribed hexagon, move with equal velocities, in directions inclined at angles of 30° and 60°, respectively, to their planes. Compare the resistances perpendicular to their motions.

11. Sum the following series to n terms,

$$r - \frac{r^{\frac{3}{2}}}{2} + \frac{r^2}{4} - \&c.$$

and $1 . 2 + 2 . 5 + 3 . 8 + 4 . 11 + \&c.$

12. Find the fluents of

$$\frac{p\dot{x}}{a - bx^2}, \text{ and } \frac{x^3\dot{x}}{\sqrt{(a - x)}}.$$

13. Having given the ratio of the periodic times in two circles, described about different centres of force situated in their centres, and also the ratio of the radii, it is required to find the ratio of the absolute forces.

14. Determine the angle between the apsides in an orbit nearly circular, the force being constant; taking an ellipse about the centre for the revolving orbit.

TUESDAY AFTERNOON.—MR. PEACOCK.

Third and Fourth Classes.

1. If $\frac{p_0}{q_0}$ and $\frac{p_1}{q_1}$ be two consecutive terms in a series of fractions converging towards $\frac{a}{b}$; then

$$p_0 q_1 - p_1 q_0 = \pm 1.$$

2. Explain what is meant by the *conjugate points* of curve lines.

3. If $u = f\left\{x, y\right\}$, shew that

$$\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}.$$

4. Find the integral or fluent of

$$\frac{dx}{x^3 - 7x^2 + 12x}.$$

5. If a body be projected perpendicularly upwards with a velocity (a), its height (x), at the end of the time (t), is determined from the equation

$$(a - 2mt)^2 = 4m\left(\frac{a^2}{4m} - x\right).$$

6. Enumerate the different practical methods of determining the latitude of a ship at sea.

7. Explain the method of measuring altitudes, by means of the barometer and thermometer.

8. A given rectilinear object is placed before a spherical reflector of given radius. Find the equation to the conic section which is its image.

9. Find an expression for the whole time of descent of a body from a distance (a) to the centre of force, when the force varies inversely as the square of the distance.

10. Mention some of the problems, upon which the trisection of an angle, by common Geometry, may be made to depend.

TUESDAY EVENING.—MR. PEACOCK.

1. Demonstrate the rule for the extraction of the square root in numbers.

2. Every prime number of the form $4n + 1$ is the sum of two squares.

3. Approximate to the value of x in the equation

$$x^3 - 2x - 5 = 0,$$

and explain the defects of the methods of approximation, as given by Newton and Ralphson.

4. Prove that

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} \cdot u_{x+n-1} + \frac{n(n-1)}{1 \cdot 2} u_{x+n-2} - \&c.$$

5. Integrate the differentials

$$\frac{dx}{x^5 \sqrt{\left\{1 + x^2\right\}}}, \quad \frac{a^x\, dx}{x^2}, \text{ and } dx \cos^2 x \sin^3 x.$$

6. Integrate the differences or increments

$$x^3 \text{ and } e^x \cos x\,\theta.$$

7. Integrate the differential equations

(1) $\dfrac{d^2 y}{dx^2} = \dfrac{m}{(a - y)^2}$.

(2) $d^2 y + \text{A}\, y\, dx^2 = \text{X}\, dx^2$, where X is a function of x.

(3) $\dfrac{dz}{dx} - \dfrac{y}{x}\dfrac{dz}{dy} = -\dfrac{y^2}{x^2}$.

8. Integrate the equation of differences,

$$u_{x+2} - \text{A}\, u_{x+1} + \text{B}\, u_x = 0.$$

9. Given the length of the curve; required its nature when its centre of gravity is most remote from the axis.

10. If two lines intersect each other within a parabola, the ratio of the rectangles contained by their respective segments will be the same with the ratio of the rectangles made by the segments of any other two lines which intersect each other, and which are respectively parallel to the former.

11. Apply D'Alembert's principle to the determination of the distance of the centres of oscillation and suspension in a compound pendulum.

12. A triangular prism being immersed in a fluid of greater specific gravity than itself, it is required to determine the different positions in which it will rest in equilibrium.

13. A machine, driven by the impulse of a stream, produces the greatest effect when the wheel moves with one-third of the velocity of the water.

14. At a place whose latitude is 48°. 50′. 14″, the meridian altitude of the sun's upper limb was observed to be 62°. 29′. 56″; it is required to determine the sun's declination, the refraction being 29″, the sun's parallax and apparent diameter of their mean values, and the sine of 27°. 30′. 4″ = .4617.

15. Explain the method of correcting an error in the longitude of a place, by means of the occultation of a given fixed star by the moon.

16. If r be the radius of an isosceles lens, whose focal length is equal to that of a lens whose radii are r_1 and r_2; then

$$\frac{1}{r} = \frac{1}{2}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\}$$

17. If D be the length of a degree of the meridian at a point whose latitude is λ, Δ the length of a degree of a curve perpendicular to the meridian at that point, a the axis major of the meridian, and e the difference of the semi-axes; then

$$\frac{e}{a} = \frac{\Delta - D}{2\,\Delta \cos^2\lambda} \text{ (nearly).}$$

18. The moon is retained in her orbit by the force of gravity.
Newton. Lib. III. Prop. 4.

19. The sum of the sides of a right-angled triangle remaining the same, required the nature of the curve to which the hypothenuse is always a tangent.

20. Explain the method of drawing a normal to a given curve surface.

21. Give an account of the controversy between the followers of Newton and Leibnitz, concerning the measure of motion, and reconcile the experiments and results to which the latter appealed, with the measure assumed by the former.

22. If two chords of a circle intersect each other at right angles, the sum of the squares described upon the four segments is equal to the square described upon the diameter.

23. Give some account of the Analysis of the Ancient Geometers. Exemplify it in the solution of the following problem: "To bisect a triangle by a straight line drawn through a given point in one of its sides."

CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the degrees of B. A. in January, 1818.

BY THE TWO MODERATORS.

MONDAY, JANUARY 20, 1818.

MONDAY MORNING.—MR. FRENCH.

First and Second Classes.

1. The present value of an annuity, to continue for a term of years at a given rate of compound interest, $= m \times$ the present value of the same annuity, to be paid only during the latter half of the same term; required to find when the annuity will cease?

2. To determine the numerical value of the arc A which will *satisfy* the following equation:

$$\sin. B + \sin. (A - B) + \sin. (2A + B) = \sin. (A + B) + \sin. (2A - B).$$

3. Prove that the sum of all the coefficients of a binomial raised to the $(2n)^{th}$ power: the coefficient of its middle term :: 2 . 4 . 6 . &c. to n factors : 1 . 3 . 5 . &c. to n factors.

4. A body is suspended from a given point in the horizontal plane, by a string of known length, which is thrust out of its vertical position by a rod (supposed without weight) acting from a given point in the plane, against the body; shew that the tension of the string varies inversely as the tangent of the inclination of the rod to the horizon.

5. Two equal hollow paraboloids have a common axis, which is vertical, and such a quantity of water is poured in between them, as just to touch the lowest point of the inner figure; demonstrate that the surface of the water will be a tangent plane to this figure, in any position of the common axis.

6. In Gregory's telescope, the focal length of the larger reflector, the position and focal length of the eye-glass, and the distance between the two images of a remote object being given; required to find the position and focal length of the smaller re-

flector, which will cause the telescope to magnify the object in any proposed ratio.

7. Having given $nt = u - e . \sin . u$; required the first four terms of the series expressing n in terms of nt?

8. A spherical body descends in a fluid by gravity; to determine the quantity of the resistance, when the body has described a given space.

9. The force varying inversely as $(\text{dist})^5$ and the velocity being that which would be acquired from infinity, a body is projected from an apse; compare the time of its descent to the centre with the periodic time in a circle, whose radius = half the apsidal distance.

10. To find the fluent of

$$\frac{\dot{x}\sqrt{(1-x^2)}}{(1+x)^2}.$$

11. Sum the following series

$$\frac{1}{1 \cdot 1} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 9} + \text{\&c. in inf.}$$

12. A body described by a parabola about the focus, and at same time the figure moves uniformly in a direction *perpendicular* to its axis, which continues parallel to itself; to determine the path described by the body in fixed space.

MONDAY AFTERNOON.—MR. FRENCH.

Fifth and Sixth Classes.

1. $\frac{x}{x+2} - \frac{x-9}{3x-20} = \frac{9}{13}$. Required the values of x?

2. Required the ratio which is one half of the ratio $(\sqrt{(32)} : 25)$.

3. The sum of an arithmetic series is 5, the first term 11, and the common difference -5; find the number of terms.

4. To determine the value of tan. 30° to two places of decimals. (rad. = 10000).

5. P being any point in an ellipse, whose semi-axis major is AC, prove, that, if the normal (PG) be produced to meet the conjugate diameter in F and the minor axis in V,

$$PF . PV = (AC)^2.$$

6. Two circles touch each other internally, and the area of the *lune* cut out of the larger is equal to twice the area of the smaller circle. Required the ratio of the diameters of these circles.

7. A body is projected perpendicularly upward with a velocity of 64 feet per second, find the time of ascent through 63 feet.

8. The length of the gage of a condenser is 12 inches, and the space occupied by the air in it, after two descents of the sucker, is half its whole length; to determine the space which the air will occupy after the third descent of the sucker.

9. Having given the focus of incidence of a pencil of rays which passes nearly perpendicularly through the sides of a prism, and sin. I : sin. R :: n : 1 out of the ambient medium into the prism; required (n being a proper fraction) to find the focus of emergent rays.

10. If a star be situated nearer to the pole of the ecliptic than to that of the equinoctial, shew that its right ascension exceeds 180°.

11. A body is revolving in a given circle about its centre, if the absolute central force be increased in a given ratio, what change must be made in the velocity of the body that it may still describe the same circle?

12. Demonstrate, as Newton has done (Cor. 2. Prop. 10.), that the periodic times in all ellipses about the same centre are equal.

13. Assuming the velocity to vary as $\frac{\sqrt{(a-x)}}{\sqrt{(x)}}$, a being the initial distance, and x the variable distance of the body from the centre of force; to determine the law of the centripetal force.

14. One root of the equation $(x^3 - 11x^2 + 37x - 35 = 0)$ is $3 + \sqrt{2}$; required the remaining roots.

15. Required the fluxion of $\frac{y\sqrt{(a^2 - y^2)}}{a^3 - y^3}$.

16. Find the fluent of $\frac{rz^2\dot{z}}{e - fz}$.

MONDAY AFTERNOON.—MR. FALLOWS.

Third and Fourth Classes.

1. What part of half-a-crown is equal to $\frac{3}{7}$ of 1*s*.5$\frac{1}{2}$*d*.?

2. Of all triangles under a given perimeter, and a determinate side, shew that to be the greatest in which the two indeterminate sides are equal.

3. If the p^{th} and q^{th} terms of an arithmetical progression be P and Q, find the sum of n terms of the series.

4. Transform the cubic equation $x^3 + px^2 + qx + r = 0$, whose roots are a, b, c, into another whose roots are

$$\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \left(\frac{1}{a^2} + \frac{1}{c^2}\right), \left(\frac{1}{b^2} + \frac{1}{c^2}\right).$$

5. A ship sails directly north at the rate of (a) miles an hour, and the velocity of the wind is (b) miles an hour; find the direction of the wind so that the vane may point due west.

6. Find the quantity of water discharged from a small given orifice in the side or bottom of a vessel in a given time; the vessel being kept constantly full.

7. Having given the radius of an arc of any colour in the secondary rainbow, find the ratio of the sine of incidence to the sine of refraction when rays of that colour pass out of air into water.

8. If a body revolve in an ellipse (whose major and minor axes are given) with the force tending to its focus, and the time of revolution be given; find the actual velocity of the body at any given point in its orbit.

9. If the hyp. log. $\dfrac{\sqrt{(a^2 + x^2)} + a}{\sqrt{(a^2 + x^2)} - a} = b$, find x.

10. Find the surface of the solid generated by the revolution of a common cycloid about its axis.

11. Explain why the effect of aberration on a star not situated in the solstitial colure at six o'clock, either evening or morning, is partly in declination, and partly in right ascension.

12. A luminous point is placed in the axis of a glass lens, which is *plane* on one side and *curved* on the other; find the nature of the curved surface so that rays diverging from the luminous point may, after passing through the lens, be refracted accurately to another given point.

13. The right ascension and declination of a star being given, as also the time of the year when it rises with the sun; find the latitude of the place.

14. The increment of the hyp. log. $(x) = 2$

$$\left\{\left(\frac{\dot{x}}{2x + \dot{x}}\right) + \frac{1}{3}\left(\frac{\dot{x}}{2x + \dot{x}}\right)^3 + \frac{1}{5}\left(\frac{\dot{x}}{2x + \dot{x}}\right)^5 + \&c.\right\}$$

15. Find the following fluents:

$\displaystyle\int\frac{x^2\dot{x}}{(a^2 + x^2)^2}$; $\displaystyle\int\frac{a^x\dot{x}}{\sqrt{(1 - a^{2x})}}$; $\displaystyle\int z\dot{x}$, where z is an arc whose tangent $= x$.

MONDAY EVENING.—MR. FALLOWS.

1. If £1. £8. £27. &c. be lent at the beginning of the first, second, third, &c. year; find the whole amount due, at simple interest, at the end of n years, at r rate per cent.

2. Given the four sides of a quadrilateral figure inscribed in a circle; to find its diagonals.

3. A string passing over a fixed pulley is coiled, on each side of it, round two cylinders of equal weight (w), the one being of uniform density, the other collected in the circumference; find the tension of the string when they are at liberty to move; the inertia of the string and pulley not being taken into account.

4. Given the area of a right-angled triangle; to find the curve to which the hypothenuse is always a tangent.

5. At what angle must two plain reflectors be inclined, so that a man standing in a given position, may see his face in *profile* in the image of one of them?

6. The ages of two persons being equal; find the value of an annuity of £1. for their joint lives.

7. A body revolves in an ellipse, the force being in the focus; shew that if an additional velocity be communicated to it in its descent from the higher to the lower apse, the apsides are regressive, and if communicated in its ascent from the lower to the higher, they are progressive.

8. Two barometers whose lengths are a, a' inches, contain b, b' inches of air respectively; if on account of some change in the weather the former barometer falls one inch, what will be the depression in the latter; supposing a perfect barometer to stand at 30 inches before the depression?

9. Equal altitudes of the sun are taken before and after its passage over the meridian and the times of observation noted by a chronometer; find its error when the change of declination is taken into account.

10. Find the integral of cos. z; and from thence, sum the series cos. a+cos. $(a+b)$+cos. $(a+2b)$ . . . . . . + cos. $(a+nb)$.

11. Find the following fluents:

$$\int x^2\dot{x}\,.\,\text{arc.}\,(\sin. = x), \int\frac{\dot{x}(1+x^2)}{(1-x^2)\,.\,\sqrt{(1+x^4)}}.$$

12. Find the relation between x and y in the following equations:

$$x\dot{y} - y\dot{x} - (x^2+1)\,\dot{x} = 0;$$

$$(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}} + 2a^{\frac{1}{2}}\sqrt{(a-x)}\,.\,\dot{x}\,\ddot{y} = 0.$$

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13. If the mean density of the earth (considered as a sphere) be to the density at the surface as 1 : m; find that power of the distance from its centre according to which the density of its parts varies.

14. Explain the construction of *Mercator*'s Chart, and from thence find the distance of two places projected on the chart whose latitudes and longitudes are given.

15. If a prolate spheroid be cut by a plane passing through the focus, the section will be an ellipse having its focus in the focus of the spheroid.

16. Prove that the sum of the terms of *Taylor*'s Series commencing with any given term, can always be rendered less than the term immediately preceding it.

17. If a small pencil of parallel homogeneal rays be refracted into a sphere of water, and emerge parallel; shew that after two refractions and one reflection, the angle contained between the incident and emergent rays is a *maximum*, and after two refractions and two reflections it is a *minimum*.

18. A circle has the greatest triangle inscribed in it, a circle is inscribed in the triangle which has the greatest triangle inscribed in it, and so on; find the sum of all the circles and triangles.

19. If $F \propto \frac{1}{D^3}$ and be attractive; shew that six different kinds of orbits may be described with proper velocities and angles of projection, and only six; and when repulsive, only one.

20. A paraboloid rests upon a horizontal plane with its axis vertical and vertex downwards: What must be the length of its axis in order that the equilibrium may be that of *indifference?*

21. If the resistance of the medium vary partly in the simple and partly in the duplicate ratio of the velocity, and a body urged by the force of gravity ascend or descend in the medium; shew how the spaces described by the body in different times may be compared. *Newton*, Prop. 14. Book. II.

22. A rigid prismatic bar of uniform density and given length is placed in the straight line joining two centres of force, whose distance is given, and whose intensities are in the ratio of 2 to 1; find the position of the bar so that it may rest in equilibrium, supposing $F \propto \frac{1}{D}$.

23. The lunar orbit being supposed circular; compare the moon's velocity in quadratures with its velocity at any given place of its orbit, taking into consideration that the earth and moon revolve about their common centre of gravity. *Newton*, Prop. 26. Book III.

24. Investigate the following formula for clearing the moon's distance: $\text{ver. sin. }(\mathrm{D}) = \text{ver. sin. }(\mathrm{A} - \mathrm{B}) + \frac{\cos.\ \mathrm{A} \cos.\ \mathrm{B}}{\cos.\ a \cos.\ b} \left\{ \text{ver. sin. }(d) - \text{ver. sin. }(a - b) \right\}$ where A, B; a, b, are the true and apparent altitudes; D, d the true and apparent distances.

TUESDAY MORNING.—Mr. FALLOWS.

First and Second Classes.

1. Find two fractions whose denominators are prime to each other and their sum $\frac{23}{45}$.

2. The area of a trapezium is equal to the product of its two diagonals multiplied by half the sine of the angle formed by their intersection.

3. In the expansion of $\frac{a + bx + cx^2}{1 - \alpha x - \beta x^2 - \gamma x^3}$; find the general term.

4. Given the lengths of two ordinates of the logarithmic curve and the portion of the abscissa intercepted between them: to construct the curve.

5. Find the *position* of the centre of gravity of any number of bodies situated in different planes.

6. If a body fall by the action of an uniform force and describe (a) and (b) feet in the m^{th} and n^{th} second respectively, (reckoning from the beginning of the motion); find the space described in the x^{th} second.

7. Two given glass meniscuses of the same diameter and the same focal length being joined together with their convex sides outward, and the included space being filled with water; find the focal length of the lens; its thickness not being considered.

8. If an oblate spheroid whose axes are given, be filled with water and placed with its major axis perpendicular to the horizon: find the time of emptying through a small given orifice at the extremity of the vertical axis

9. Prove, *strictly*, that $\dot{v} = \mathrm{F}\dot{t}$, $v\dot{v} = \mathrm{F}\dot{x}$, and $\frac{\ddot{x}}{\dot{t}^2} = \mathrm{F}$.

10. In a given latitude and longitude, a vertical plane declines (a°) from the south towards the west; find the place to whose horizon the plane is parallel.

11. If a body fall from rest through a given space AB towards

a given centre of force C, in t seconds; compare the force A with gravity, supposing $F \propto \frac{1}{D^3}$.

12. Investigate the nature of the curve, in which lines drawn from a given point perpendicular to the tangent may always be equal.

13. Find the integral of $\frac{x}{ax}$.

14. If an elastic chord of uniform density, whose length is (L) and weight (W), be stretched in an horizontal position by a given weight (w) and the increment of length be (l); find the length of the chord when suspended by one of its extremities; the increment of its length being always as the weight which stretches it.

TUESDAY AFTERNOON.—MR. FALLOWS.

Fifth and Sixth Classes.

1. A person sold goods to the value of £1000 and gained 20 per cent. What was the prime cost?

2. Surd roots of the form $\pm\sqrt{(b)}$ enter equations by pairs.

3. If two triangles are to each other as their bases; prove that they have the same altitude.

4. A body is projected up a plane inclined to the horizon at an angle of 30° with a velocity of 20 feet per second, find where it will be at the end of four seconds.

5. If $\sin. (A - B) = \frac{1}{2}$ rad. and $\sin. (A - B) = \cos. (A + B)$ find A and B.

6. Find how far a body will fall from rest; while a pendulum whose length is 20 inches makes 10 vibrations.

7. Define logarithms, and shew from the definition that $\log. (ab) = \log. a + \log. b$; $\log. \left(\frac{a}{b}\right) = \log. a - \log. b$; $\log. a^n = n \log. a$.

8. A hollow globe is filled with fluid; compare the internal pressure with the weight of the fluid.

9. In the magic lantern, prove that no image will be formed upon the screen, unless the distance between the lantern and the screen be greater than four times the focal length of the lens.

10. The sun is at the same altitude at equal intervals of time before and after its passage over the meridian, supposing no change in declination to have taken place during the interval.

11. $F \propto \frac{1}{D^2}$; a body revolving in a circle at a given distance from the centre will by its motion at any point turned upwards ascend to twice its distance from the centre.

12. Find the following fluents:

$$\int \frac{a\dot{x}}{b + \frac{c}{x}}, \int \frac{x\dot{x}}{(1 + x^4)^{\frac{3}{2}}}.$$

13. The circumference of a circle to its diameter is nearly in the ratio of 22 to 7.

14. Inscribe the greatest parallelopiped in a sphere.

15. Every inscribed triangle formed by any tangent and the two intercepted parts of the asymptotes of a hyperbola, is equal to a given area.

16. Find the radius of curvature at the vertex of a common parabola.

17. If a body revolve in a logarithmic spiral, find the law of centripetal force tending to the pole of the spiral.

TUESDAY AFTERNOON.—MR. FRENCH.

Third and Fourth Classes.

1. If A, B, C, be three angles of a plane triangle, having given $\cos. B = \frac{1}{2} \frac{\sin. A}{\sin. C}$; prove the triangle to be isosceles.

2. Prove that the arc $\frac{60^\circ}{2^{12}} = \left(2^2 - \frac{1}{2^2}\right)^3$ seconds $= 52''. 44'''. 3^{iv}. 45^{v}$.

3. If a body be projected perpendicularly upward, the time of its ascent through any space is determined from the quadratic equation $(mT^2 - V . T + S = 0)$; shew that the least root is that which answers the conditions of the problem.

4. If a given pendulum be made to oscillate in a cycloid; its greatest velocity in the cycloidal arc : its greatest velocity in the circle :: the cycloidal arc described in its descent : the chord of the circular arc described.

5. A solid of revolution, whose axis is perpendicular to the horizon, empties itself by a small given orifice; required its nature, when the velocity of the descending surface varies inversely as the ordinate of the generating figure.

6. An eye being placed so as just to see the lowest point of an hemispherical vessel, when empty; it is required to determine the perpendicular depth of that point of its inner surface

nearest to the eye, which is brought into view when the vessel is filled with water.

7. To a spectator in the northern hemisphere, the sun, whose declination = 15°, rises just two hours before noon; prove that tan. latitude of the place of observation = $\frac{1}{2}\sqrt{(3)} \cdot \sqrt{\frac{(1 + \frac{1}{2}\sqrt{(3)})}{1 - \frac{1}{2}\sqrt{(3)}}}$. (rad. = 1).

8. A cylinder, whose weight = 133.6℔. and rad = 10, revolves about its horizontal axis; to determine the time in which a weight of 20℔. acting by means of a string at the circumference of the cylinder, will generate a velocity of 1 foot per second at a distance = 1 from the axis. (m = 16 feet).

9. If bodies move in a logarithmic spiral from different points to its pole, shew that the times of their motion are as the squares of the spaces which they respectively describe.

10. According to what power of the distance must the force vary, that the areas, *dato tempore*, in all circles uniformly described about the centre of force, may be equal?

11. If the point A (Prop. 41. Sect. 8.) be removed to an infinite distance from the centre of force, shew from *Newton's* construction (Cor. 3.), that the hyperbolic spiral will become a circle.

12. The length of the catenary = $a\left(e^{\frac{\text{D}}{a}} + e^{\frac{\text{D}}{-a}}\right)$, D being its greatest ordinate and a the lateral tension. Required a proof.

13. $y = \frac{x \cdot \sqrt{(1 - x^2)}}{\sqrt{(1 - a^2x^2)}}$. Required the maximum value of y.

TUESDAY EVENING.—Mr. FRENCH.

1. A person transfers £1000 stock from the *five per cents.* to the *three per cents.* when the former are at 110, and the latter at 84; if, at the end of six months, the *five per cents.* have risen to 112, what must then be the price of the *three per cents.* that he may sell out without having gained or lost by the transfer?

2. Having given two distances from the focus of a parabola and the angle between them, to construct the parabola.

3. To determine the greatest straight line which can be drawn from a given point in the minor axis of an ellipse to its periphery.

4. A ball is projected from a given point in the horizontal

plane at an angle of 30°, and after describing two-thirds of its horizontal range, strikes against a sonorous body; having given the whole interval between the instant of projection, and the instant when the sound reaches the point of projection, to find the initial velocity.

5. The periodic times of four bodies being 24, 22, 20 and 18 days, respectively; in what time, after leaving a conjunction, will they all be again in conjuction, and what number of revolutions will each have performed?

6. If rays fall nearly perpendicularly upon a spherical refracting surface of a denser medium converging to a point between the surface and its centre, and sin. I : sin. R :: $m : n$; shew that the greatest distance between the conjugate loci $= \frac{\sqrt{m} - \sqrt{n}}{\sqrt{m} + \sqrt{n}} \cdot r$. ($r$ being the radius of the refracting surface.)

7. The values of an ounce of platina, gold, and silver being p, g and s respectively, and their specific gravities a, b, c; compare the value of a coin, made of platina and silver, and which equals a guinea in weight and magnitude, with the value of a guinea.

8. Shew that the n^{th} term in the series of hexagonal numbers is the same with the $(2n - 1)^{\text{th}}$ term in the series of triagonal numbers.

9. The point C is such that all straight lines drawn from it to two given points A, B, are in a given ratio; prove that the *locus* of C is the circumference of a circle.

10. A small object is placed at such a point in the diameter of a sphere of water as to be distinctly seen, after one reflection and one refraction, by an eye in that diameter produced; compare its visual angle with the visual angle of the same object when placed in the principal focus of the sphere.

11. Find the following fluents;

$$\int \frac{a\dot{x}}{\sqrt{(1 - x^4)}}, \text{ when } x = 1; \int \frac{(a^2 + x^2)^m \cdot \dot{x}}{(\text{hyp. log. } x)^n}.$$

12. A wheel, in 36 revolutions, passes over 29 yards, and in x of these revolutions it describes $\overset{\text{yds.}}{z} + \overset{\text{feet}}{y} + \overset{\text{inches}}{5}$; to find the values of x, y and z.

13. To find the place of a body in an elliptic trajectory at any given time. (*Newton*, Vol. I. Sect. 6.).

14. Deduce Kepler's law of the equable description of areas about the centre of force from the three fluxional equations of motion.

15. Investigate the expression for the precession in right ascension of a star, whose right ascension is greater than 180° and less than 270°.

16. Required the sum of the terms of a binomial $(a+x)^m$, at intervals of n from each other, beginning with the $(p+1)^{\text{th}}$ term.

17. A given hemispherical vessel, whose thickness is t, resting upon its base, is filled with fluid to a depth = half of its inner radius; required the ratio of the specific gravities of the vessel and the fluid, when the vertical pressure of the fluid = the weight of the vessel.

18. To resolve $(a^2 - ab \,.\, 2 \cos.\, \theta + b^2)^{-2s}$ into a series of cosines of arcs, the multiples of θ, by means of the formula $2 \cos.\, m\theta = x^m + \frac{1}{x^m}$, and the binomial theorem.

19. A body moves in a logarithmic spiral, the centripetal force varying inversely as $(\text{Dist.})^2$, and the resistance as the density of the medium and the square of the velocity jointly; from these data determine the law of the density.

20. Sum the following series:

$$\frac{8}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{16}{5 \cdot 7 \cdot 9 \cdot 11} + \frac{24}{9 \cdot 11 \cdot 13 \cdot 15} + \text{\&c. to } n \text{ terms}$$

by increments,

$$ax^{\alpha} + (a+b)x^{\alpha+\beta} + (a+2b)x^{\alpha+2\beta} + \text{\&c. to } n \text{ terms.}$$

21. If seven balls be drawn from a bag containing eleven in all, five of which are white and six black; what is the probability that three white balls will be drawn?

22. Prove that the sum of all the numbers of n places, which can be formed with the n digits a, b, c, &c. : sum of all the numbers of n places which can be formed with the n digits p, q, r, &c. of the same scale :: $a+b+c+$ &c. : $p+q+r+$ &c.

23. In a revolving fluid spheroid of small eccentricity, shew that, if $\sin.^2$ lat. $= \frac{1}{3}$, the distance from the centre (CP) = the radius of an equi-capacious sphere, and that the central attraction of P arising from the mutual gravitation of the particles of the spheroid, is equal to its attraction to the same sphere at rest.

24. ABCDE is a pentagonal billiard table, with unequal but given sides and angles; it is required to find that point in one of its sides, and the direction of impact, such, that an elastic ball may continually describe the same path, striking every side of the table in succession.

CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the degrees of B. A. in January, 1819.

BY THE TWO MODERATORS.

Monday, January 18, 1819.

Monday Morning.—Mr. Peacock.

First and Second Classes.

1. What number of degrees, minutes and seconds are contained in an arc equal to radius?

2. If from a point without a parallelogram, lines be drawn to the extremities of two adjacent sides and of the diagonal which they include; of the triangles thus formed, that, whose base is the diagonal, is equal to the sum of the other two.

3. If Mx^{n-m} be the first negative term of the equation

$$x^n + px^{n-1} + \ldots - Mx^{n-m} - \ldots = 0.$$

and if P be the greatest negative coefficient, then $1 + \sqrt[m]{P}$ is greater then the greatest root of the equation.

4. If the inverse ratio of any two consecutive coefficients of the series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.$$

be finite, it is always possible to assume x so small, that any one term of the series may exceed the sum of all those which follow it.

5. In the direct collision of bodies, the velocity of the centre of gravity is the same before and after impact.

6. The bulb of a thermometer is successively plunged into boiling water and melting ice, and the mercury in the tube falls a inches: given the diameter of the tube, and the diminution of bulk due to one degree of temperature, to find the capacity of the bulb.

7. If rays nearly parallel, are incident upon a concave spherical reflector, whose radius is r, and if d and d' be the distances of the foci of incident and reflected rays, then

$$\frac{1}{d} + \frac{1}{d'} = \frac{2}{r}.$$

8. Explain what is meant by the line of *collimation;* and shew by what means any error arising from it, may be compensated in the circular transit instrument with an azimuth motion,

9. Explain the method of finding the longitude, by observing the increase of the moon's right ascension, in the interval of her transit over two meridians.

10. Two lines AP and BP in the same vertical plane, pass through two points A and B situated in the same horizontal line: find the locus of the point P, so that the time of a body's descending down AP and ascending up BP with the velocity acquired, may be constantly the same.

11. Integrate the differential equation

$$e^x dx - \frac{y\,dy}{e^x} = dy - y\,dx.$$

12. All epicycloids, the radii of whose generating circles bear an assignable numerical ratio to the radii of their bases, are expressible by finite algebraical equations.

13 The cycloid is the curve of quickest descent, between two points which are not in the same vertical line; demonstrate this by the Calculus of Variations.

MONDAY AFTERNOON.—MR. PEACOCK.

Fifth and Sixth Classes.

1. What is the purchase money of £156. 15*s*. 1*d*. 3 per cent. annuities, at $74\frac{1}{2}$ per cent?

2. Give the reason why quadratic equations admit of two solutions.

3. Investigate an expression for the number of combinations of n things, taken m and m together.

4. Explain in what case and for what reason, *Cardan*'s formula for the solution of a cubic equation, does not enable us to determine the roots.

5. Sum the series.

$$\frac{1}{\sqrt{2}\,(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})\,(2+\sqrt{2})} + \frac{1}{(2+\sqrt{2})\,(3+\sqrt{2})} + \text{\&c. } \textit{in infinitum.}$$

6. Prove that if $2 \cos A = x + \frac{1}{x}$, then $2 \cos mA = x^m + \frac{1}{x^m}$.

7. Explain the method of determining the height of an inaccessible object; give the formulæ of solution of the triangles and adapt them to logarithmic computation.

8. The lines drawn from the angles of a triangle, to the bisections of the opposite sides, all meet in one point.

9. A body descends 400 ft. down a plane inclined at an angle of 30°; Calculate the actual time of descent to 3 places of decimals.

10. If W be the weight sustained by the wheels of a carriage; what is the force necessary to keep it at rest, upon a road inclined at a given angle to the horizon, the line of draught being parallel to the road?

11. Explain fully the construction and principle of the common pump.

12. The periodic times of the planetary bodies are independent of the eccentricities of their orbits.

13. Explain the phases of Venus.

14. What is the cause of twilight? Within what limits of polar distance, is there at least one day of the year, when it will continue all night?

15. When parallel rays are incident nearly perpendicularly upon a spherical refracting surface, find the geometrical locus of refracted rays.

16. Investigate the rule for finding the *maxima* and *minima* values of a function of one variable, and shew in what manner they are distinguished from each other.

17. Find an expression for the radius of curvature of the ellipse.

18. Find the centre of gravity of the arc of a cycloid.

19. In the collision of perfectly elastic bodies the relative velocity is the same before and after impact.

20. Given the weight of a body in water and in air, to find its true weight.

21. Compare the forces by which the Moon is attracted by the Earth and Sun.

MONDAY AFTERNOON.—MR. GWATKIN.

Third and Fourth Classes.

1. Extract the square root of $\frac{a^2 c}{b} - cf + 2ac\sqrt{\left(-\frac{f}{b}\right)}$.

2. Solve the equation $\frac{\sqrt{(a + b x^n)} - \sqrt{(a)}}{\sqrt{(b x^n)}} = c$, and find x and y from the following

$$\left.\begin{array}{l} x^4 - x^2 + y^4 - y^2 = 84 \\ x^2 + x^2 y^2 + y^2 = 49 \end{array}\right\}.$$

3. Produce a given straight line so that the rectangle under the given line, and the whole line produced may equal the square of the part produced.

4. Find by the method of continued fractions a series of fractions converging to $\sqrt{3}$.

5. Prove that the third term of the equation $x^3 - px^2 + qx - r = 0$, cannot be taken away if p^2 be less than $3q$.

6. P and Q sustain each other on two inclined planes, which have a common altitude, by means of a string parallel to the planes. Shew from geometrical as well as mechanical considerations that if they be put in motion, their centre of gravity describes a right line parallel to the horizon.

7. Bisect the arc of a semicycloid; and if a body oscillate through it, compare the times of describing the first and last half.

8. A right cone whose axis is vertical is just immersed in a fluid, first with its base, then with its vertex downward. Compare the pressure on its whole surface in each case.

9. An object being placed between two plane reflectors inclined at the angle 22°. 30′, find the number of images, and shew that two of them coincide.

10. The whole disk of the moon is faintly visible when she is near conjunction, and also when suffering a total eclipse. Explain these phenomena.

11. Find the fluxion of arc whose tang. $= \sqrt{\left(\frac{1 - x}{1 + x}\right)}$, and shew that $\int dx\, (1 - x^2)^{\frac{2n-1}{2}}$ (taken between the limits of $x = 0$ and $x = 1$) $=$

$$\frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n - 1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n} \cdot \frac{\pi}{2}.$$

12. Find the area of the curve traced out by the intersection of the sine of an arc, and the secant of half the arc, while the arc increases from 0 to a quadrant.

13. Shew that the number of primes is infinite.

14. Find the polar equation to the ellipse, the centre being considered the pole.

15. Supposing the density of the air to vary as the compressing force and gravity inversely as $(\text{dist.})^2$ from the earth's centre; find the density at any altitude, and shew from the result that the first of the above hypotheses is inadmissible.

MONDAY EVENING.—MR. GWATKIN.

1. Extract the square root of $14 + 8\sqrt{3}$.

2. Given the first and last terms, and the sum of an arithmetic series, to find the common difference.

3. If three straight lines not in the same plane are equal and parallel, shew that the triangles formed by joining their adjacent extremities are equal and their planes parallel.

4. Shew that the convex surface of a spherical segment is equal to the area of a circle whose radius is the distance from the pole to the circumference of its base.

5. The bodies A, B, C are acted on in parallel directions by the accelerating forces a, b, c; Find the point on which, if connected, they would balance.

6. Define a mean solar year, an apparent solar year, an anomalistic year and a sidereal year. Explain whence arises the difference between the two first, and write down the three last in order of their length.

7. With a single die, find the chance of throwing the six faces in six trials.

8. Given the base of a triangle, and the exterior angle always equal to three times the interior and opposite angle at the base, required the area of the curve which is the locus of the vertex.

9. Find the principal focus of a concavo convex lens of inconsiderable thickness.

10. If a hemispheroid and a paraboloid have the same base and altitude, shew that their solid contents are as 4 : 3.

11. A paraboloid of given dimensions and specific gravity floats with its axis vertical on a fluid whose specific gravity is known. How far may the axis be increased before it tends to fall from its vertical position.

12. If the difference of two numbers be invariable, shew that as those numbers increase the difference of their logarithms diminishes.

13. Integrate the quantities,

$$\frac{dx}{(bx + cx^2)^2}, \quad \cos.^2 x \,.\, e_x \, dx, \quad \frac{dx}{\sqrt{\{a + x - \sqrt{(a^2 + x^2)}\}}},$$

and shew that $\int \frac{d x}{a + b . \cos x} = \frac{1}{\sqrt{(a^2 - b^2)}} \text{arc cos} \frac{b + a . \cos . x}{a + b . \cos . x}$, a being greater than b.

14. Two planes equal in length are inclined at 45° and 30° to the horizon. A body is projected downward from the top of the first, and another upward from the bottom of the second, each with the velocity acquired down a vertical line equal in length to either plane. Compare the times of describing each plane, and the velocities at the end of the motion.

15. Shew that Newton's trochoid in the sixth section has a point of contrary flexure, and find its position.

16. Find the length of the meridian for any latitude in Mercator's chart, the oblate figure of the earth being considered

17. Prove that, in the orbit described round the sun by the centre of gravity of the earth and moon, the elliptical form and the equable description of areas are much more nearly preserved than in that which the earth itself describes.

18. Newton, Sect. 9. Prop. 44. Find the ultimate intersection of Cp the radius vector of the moveable orbit and of the line $m n$ which measures the differential force.

19. Integrate the equations,

$$\sqrt{x} . d y = \sqrt{y} . d x + \sqrt{y} . d y ;$$

$$x \frac{dz}{dx} + y \frac{dz}{dy} = n \sqrt{(x^2 + y^2)}.$$

20: Define the circle of curvature, and *thence* deduce the expressions for its radius and co-ördinates of the centre. Determine whether the circle of curvature cuts the curve at the point of contact or merely touches it ; and apply your result to the case of the ellipse at any point and at the extremities of the semi-axes.

21. The earth being supposed spherical and all its matter collected in the surface, in which a circular aperture of given radius is made, and from whose middle point a body being let fall descends to the centre of the earth, find the velocity acquired at any point of the descent.

22. Explain what is meant by the particular solution of a differential equation, and how it arises. Give the method of deducing it, first from the complete integral, and next from the differential equation ; and shew that the results thus obtained coincide.

23. Point out the method of determining the max. and min. values of an expression containing two variables; and give the criterion which decides whether the value thus obtained is a maximum, a minimum, or neither.

24. Shew that the planes of the circles which measure the greatest and least curvature of a surface at any point are at right angles to each other; and having given the radii of these determine the rádius of curvature in a plane which is inclined at any angle to the former.

TUESDAY MORNING.—MR. GWATKIN.

First and Second Classes.

1. Find the price of a marble slab 5ft. 7in. long, and 3ft. 5in. wide, at 6*s*. per square foot.

2. Construct a tetrahedron upon a given straight line, and find the radius of the sphere described about it.

3. A fraction in its lowest terms whose denominator is prime to 10 produces a circulating decimal. Required proof.

4. Find the right line of quickest descent from a right line to a point, the latter line and point being given in position, but not in the same vertical plane.

5. Shew how the focus of a given parabola may be found.

6. Find the weight and magnitude of a solid by weighing it in two fluids whose specific gravities are known.

7. A small rectilineal object is placed before a spherical reflector at a given distance from it and inclined at a given angle to the axis. Required the position and inclination of the image.

8. Given the base of a triangle and ratio of the angles at the base, draw an asymptote to the curve traced out by the vertex.

9. Integrate the following expressions
$\frac{\sqrt[3]{(1-x^3)}}{x^5}dx$, $\frac{dx}{\sqrt{(A+Bx+Cx^2)}}$; and solve the equation
$x^2 d^2 y = ay\, dx^2$.

10. Force $\propto \frac{1}{(dist.)^2}$; shew, that if a particle of matter be attracted to a straight line, the direction in which it begins to move is determined by bisecting the angle formed by the lines which join the particle and the extremities of the attracting line.

11. In the expansion of $(1+x+x^2)^n$ write down the coefficient of x^n.

12. Find the centre of gyration of a cube revolving round an axis which passes through its centre of gravity.

13. Sum the series $\tan. A + \frac{1}{2}. \tan. \frac{1}{2} A + \frac{1}{4} \tan. \frac{1}{4} A +$ &c. ad infin.

14. Shew how a plane may be drawn touching the surface of any solid; and draw a plane touching in a given point the surface of an ellipsoid whose equation is

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$; x, y, z, being the co-ordinates, and a, b, c, the semi-axes.

TUESDAY AFTERNOON.—MR. GWATKIN.

Fifth and Sixth Classes.

1. Extract the square root of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$

2. Solve the equation, $\frac{a}{x} + \frac{\sqrt{(a^2 - x^2)}}{x} = \frac{x}{b}$,

and find the values of x and y in the following equations

$$x^m y^n = a, \quad x^p y^q = b.$$

3. Draw through a given point a straight line making a given angle with a given straight line.

4. A straight line can cut a circle in only two points. Required proof.

5. Trace the changes of the algebraic sign, in the sine of an arc, the tangent and secant; and explain why sec. A and sec. $(180° + A)$ which coincide should be one positive and the other negative.

6. In the direct impact of a row of perfectly elastic bodies A, B, C, &c. decreasing in magnitude, shew that the momentum communicated to each is less than that communicated to the preceding body. When is the impact of two bodies said to be direct?

7. Shew that the time in which a heavy body descends down the straight line drawn from any point in the surface of a sphere to the lowest point = the time of descent down the vertical axis of the sphere.

8. A straight line is immersed vertically in a fluid. Divide it into three portions that shall be equally pressed.

9. A straight line passes through the principal focus of a spherical reflector at right angles to the axis. Determine the conic section that forms the image. Where must the straight line be placed that its image may be a circle?

10. Given an ellipse, shew how its centre may be found.

11. $y^3 = ax^2 + x^3$. Trace out the curve. Draw an asymptote to it, and find the magnitude and position of the greatest ordinate.

12. Find the fluxion of the log. of $\frac{x}{\sqrt{(1 + x^2)}}$, and of an arc whose sine $= 2x \sqrt{(1 - x^2)}$.

13. Integrate the following expressions

$$\frac{x^4\, dx}{x^2 + a^2}, \quad \frac{x^2\, dx}{(1 - x^2)^{\frac{1}{2}}}, \text{ and } \frac{dx}{(x - a)^2 \,.\, (x - b)}.$$

14. Describe the transit instrument and adjust it to the plane of the meridian.

15. Find the centre of gravity of a spherical sector.

16. Two bodies fall to a centre of force from the same distance, one acted on by a force varying as the distance, and the other by a force $\propto \frac{1}{(\text{dist.})^2}$. The forces at first being supposed equal, compare the times of descent.

17. Given the velocity, distance, and direction of projection, when the force varies as the distance, shew that the body describes an ellipse; and find the magnitude and position of its semi-axes.

TUESDAY AFTERNOON.—MR. PEACOCK.

Third and Fourth Classes.

1. If the roots of the equation $x^3 - px + q = 0$ be real, and if we assume $\cos\theta = \frac{-q}{2} \sqrt{\frac{27}{p^3}}$, then one of the roots $=$ $2 \cos \frac{\theta}{3} . \sqrt{\frac{p}{3}}$.

2. Determine the conjugate diameters of an ellipse, which make the least angle with each other.

3. The radius of curvature is a tangent to the evolute.

4. Investigate a general expression for the co-ordinates of the centre of gravity of the area of a curve, included between a given ordinate and abscissa.

5. Given the quantities and directions of three forces acting upon a material point in different planes, to determine the quantity and direction of the resultant or compound force.

6. In the interior rainbow, the tangent of the angle of incidence is twice that of the angle of refraction.

7. A sphere of less specific gravity than water, ascends from the depth a; what is its velocity at the moment it reaches the surface?

8. Explain the method of determining the heliocentric latitude and longitude of a planet.

9. Enumerate the principal phænomena of Saturn's ring.

10. Find the centre of oscillation of a cylinder of given length and diameter, suspended by its extremity.

11. Prove, that

$$\tan n A = \frac{n \tan A - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (\tan A)^3 + \&c.}{1 - \frac{n(n-1)}{1 \cdot 2} (\tan A)^2 + \&c.}$$

12. Find the whole area of the curve whose equation is

$$a^2 y^2 - a^2 x^2 + x^4 = 0.$$

13. Find the locus of the points, in the plane of the Moon's orbit, where a body will be equally attracted by the Earth and Moon.

TUESDAY EVENING.—MR. PEACOCK.

1. If two spherical triangles have two sides of one triangle equal to two sides of the other, each to each, and the included angles equal, the triangles are equal in every respect.

2. The modulus of tabular logarithms or

$$M = 4342944819;$$

shew in what manner this number is determined.

3. It is always possible to find those roots of numerical equations, which are whole numbers or rational fractions, without the aid of formulæ of approximation.

4. Explain the method of determining the position of the nodes of the Moon's orbit: What is the physical cause of their retrograde motion?

5. The friction of a body being supposed independent of velocity, to find an expression for the time of a body's descent down a given inclined plane, the friction being equal to $\frac{1}{n}$th part of the pressure.

6. A cubical iceberg is 100 feet above the level of the sea, its sides being vertical: given the specific gravity of sea water = 1·0263 and of ice = ·9214, at the temperature of 32°, to find its dimensions. Is this position one of stable equilibrium?

7. Prove that the centres of oscillation and suspension are reciprocal. Of what use is this property, in the determination of the length of a pendulum which vibrates seconds in any given latitude?

8. Explain the method of determining the ratio of the sine of incidence and refraction both in liquid and solid bodies.

9. Given the latitudes and longitudes of two places, where the inclination of the magnetic needle is nothing, to find the point of the terrestrial equator, which is cut by the magnetic equator, supposing it a great circle of the earth.

10. Of all equal quadrilateral figures, the square has the least perimeter.

11. Integrate

$$(1.)\ \frac{dx}{x\sqrt{(bx^2-a)}} \text{ and } \frac{d\theta}{(\sin\theta)^4\cos\theta}.$$

$$(2.)\ \frac{dx}{\sqrt{(a^4-x^4)}} \text{ from } x = 0, \text{ to } = a.$$

$$(3.)\ \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{a^2}{2x}.$$

$$(4.)\ \frac{dx}{\sqrt{(1-x^2)}} + \frac{dy}{\sqrt{(1-y^2)}} = 0.$$

$$(5.)\ xy - \frac{d^2z}{dx\,dy} = 0.$$

12. Find the equation of the curve which is the locus of the extremities of the perpendiculars from the centre upon the tangents of the equilateral hyperbola, and determine the position of its tangents at the points where it cuts the axis.

13. Given

$\log 510 = 2.70757018$
$\log 511 = 2.70842090$
$\log 513 = 2.71011737$
$\log 514 = 2.71096312$

to find the logarithm of 512, by the method of interpolations.

14. Explain the principle and construction of the Achromatic Telescope.

15. What is the least velocity with which a body must be projected from the Moon, in the direction of a line joining the centres of the Earth and Moon, so that it may reach the Earth?

16. If the bulb of a thermometer be a sphere, whose diameter is 1 inch, and if the diameter of the tube be $\frac{1}{10}$th of an inch, what is the pressure upon the interior of the bulb, when the mercury stands at the altitude of 10 inches above it, exclusive of that portion of the pressure which sustains the mercury in the tube?

17. If $nt = u + e \sin u$, where u is the eccentric and nt the mean anomaly, apply *Lagrange*'s Theorem to the developement of $a(1 - e \cos u)$, in terms of cosines of nt and its multiples.

18. Prove, that in going from the equator to the pole, the increment of gravity varies very nearly as the square of the sine of the latitude. In what manner does this variation affect, 1st. the length of a pendulum vibrating seconds; and 2dly. the altitude of the barometrical column?

19. Prove, that there can be no more than five regular solids; and find the angles which their terminating planes make with each other.

20. Given the weight of the key-stone of a circular arch, in a state of perfect equilibration, and the angles formed by each of its faces with a vertical line, to find the horizontal pressure upon the abutments.

21. Prove, that

$$\tan^{-1}\frac{x}{y} = \tan^{-1}\frac{ex - y}{ey + x} + \tan^{-1}\frac{e_1 - e}{ee_1 + 1} +$$

$$\tan^{-1}\frac{e_2 - e_1}{e_1 e_2 + 1} + \ldots\ldots + \tan^{-1}\frac{e_n - e_{n-1}}{e_{n-1} e_n + 1} + \tan^{-1}\frac{1}{e_n}.$$

Where $\tan^{-1}\frac{x}{y}$ represents an arc whose tangent is $\frac{x}{y}$, and where $e, e_1, e_2, \ldots e_n$ are any numbers whatever.

22. A spherical shell with a small orifice at its lowest point, is filled with air of the density of the atmosphere, and immersed in water to a depth a: With what velocity will the water rush into the shell, and what portion of the sphere will it occupy, when the motion ceases?

23. Develope $\frac{x}{e^x - 1}$ in a series involving ascending powers of x; Of what use are the coefficients of this series in expressing the law of the coefficients of the series for $\tan\theta$ in terms of θ?

24. Enumerate, as Newton has done, the principal proofs of the truth of the theory of universal gravitation.

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